The Klein four homotopy Mackey functor structure of $H\underline{\mathbb{F}_2}$

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October 2020

A thesis submitted for the degree of Bachelor of Philosophy (Honours)
of the Australian National University

Australian National University

To Grandma Lily...

Declaration

The work in this thesis is my own except where otherwise stated.

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Acknowledgements

I want to begin by of course thanking my supervisor Vigleik Angeltveit, for this thesis would not exist if it weren't for his constant guidance, encouragement and enthusiasm for this project throughout the year. I also want to thank Ivo for his helpful comments on drafts of this thesis, and Vigleik's other honours students Jake and Roger for their support whilst we all learned this great subject together.

I am grateful for all the fantastic courses offered by this department that I've had the pleasure of taking, and for all the academics at MSI from whom I've learned so much in such a short period of time. In particular, I want to again thank Vigleik as well as Bryan Wang, James Tener, Anand Deopurkar and many more for the excellent reading courses they've run over the past couple of years. A special mention to Mircea Voineagu at UNSW who over a summer project first introduced me to the area of equivariant homotopy theory, and the idea of studying this for the group $C_2 \times C_2$ that inspired this thesis.

I would like to thank my office-mates and friends Gianni, Callum and Adam for all the shared struggles and triumphs, and for making these past few weeks not only bearable but enjoyable. Moreover, I want to thank all the other honours students for their support and encouragement in this final stretch. The community-feel among us students both at MSI and at the ANU in general has been something never to forget.

Finally, I want to thank my parents Mally and David and my brother Aaron for always being there for me, for their support during the first half of this year I spent back at home in Sydney during the pandemic, and for even driving down to Canberra to provide me with food and nourishment to get me through this intense year. Without them I would not be where I am today academically, and more importantly as a human being.

Abstract

Let $G = C_2 \times C_2$ be the Klein four group. Partial computations of the $RO(G)$-graded homotopy Mackey functors $\pi_\bigstar H\underline{\mathbb{F}_2}$ of the equivariant Eilenberg-MacLane spectrum corresponding to the constant Mackey functor $\underline{\mathbb{F}_2}$ can be found in the literature. In particular, Holler-Kriz computed in [2] the complete additive structure of the top levels $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ of the Mackey functors, and Guillou-Yarnall in [3] computed the homotopy Mackey functors graded by multiples of the regular representation ρ, namely the integer graded homotopy Mackey functors $\underline{\pi}_*(\Sigma^{k\rho} H\underline{\mathbb{F}_2})$ for each $k \in \mathbb{Z}$. In this thesis, we discuss the multiplicative structure of the top level $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ and moreover give a complete algebraic description of the homotopy Mackey functors making up $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ graded by all of $RO(G)$. Finally, we use the Bockstein spectral sequence to compute the portion of $\pi^G_\bigstar H\underline{\mathbb{Z}}$ graded by actual representations using our algebraic description of $\pi^G_\bigstar H\underline{\mathbb{F}_2}$.

Contents

Chapter 1

Introduction

A longstanding problem in algebraic topology is that of determining values of $n = 2^k - 2$ such that there exists an n-dimensional framed manifold with non-zero Kervaire invariant (necessarily then equal to one), called the Kervaire invariant one problem. This problem has now been largely settled, and we know that there exist framed n-manifolds with non-zero Kervaire invariant only for $n = 2, 6, 14, 30, 62$ and possibly 162, which is the only open case. In particular, Hill, Hopkins and Ravenel proved in the famous paper [1] that no such framed n-manifolds exist for $n \geq 254$ (i.e. for $k \geq 8$), and in doing so made vast and groundbreaking developments in equivariant stable homotopy theory.

In equivariant stable homotopy theory, we are interested in the presence of actions on our spaces and spectra by a finite group G, and we call these G-spaces and G-spectra. In doing so, the role of abelian groups in non-equivariant stable homotopy theory are replaced by Mackey functors which contain an abelian group for each subgroup of G. That is, we want to assign algebraic invariants that captures information about how our spaces or spectra behave under the action of each subgroup of G and how these relate via so-called transfer and restriction maps. Furthermore, whilst in non-equivariant stable homotopy theory these algebraic invariants are often graded over the integers $\mathbb{Z}$, we can grade our equivariant algebraic invariants over the orthogonal representation ring $RO(G)$. If X is G-spectrum, then the algebraic invariants of primary interest in this thesis are its $RO(G)$-graded homotopy Mackey functors $\underline{\pi}_\bigstar(X)$.

If X is the equivariant Eilenberg-MacLane spectrum $H\underline{M}$ corresponding to a Mackey functor $\underline{M}$, the homotopy Mackey functors $\underline{\pi}_\bigstar H\underline{M}$ can be thought of as the equivariant homology of a point with coefficients in $\underline{M}$. These computations are non-trivial, even for the cyclic group C_2 of order two. The equivariant homology of a point for the cyclic group C_8 with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$ was used in Hill, Hopkins and Ravenel's solution to the Kervaire invariant one problem. Other computations of the equivariant homology of a point for various finite groups have been done recently, for example in [2], [14], [20] and [21].

The main goal of this thesis is to completely determine the $RO(G)$-graded homotopy Mackey functors $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ for the Klein four group $G = C_2 \times C_2$, where $H\underline{\mathbb{F}_2}$ is the equivariant Eilenberg-MacLane spectrum corresponding to the constant Mackey functor $\underline{\mathbb{F}_2}$. Here $\mathbb{F}_2$ denotes the field with two elements, and one can think of the G-spectrum $H\underline{\mathbb{F}_2}$ as a G-equivariant analogue of the non-equivariant Eilenberg-MacLane spectrum $H\mathbb{F}_2$ representing singular homology and cohomology with coefficients in $\mathbb{F}_2$. The additive structure of $\pi^G_\bigstar H\underline{\mathbb{F}_2}$, where $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ denotes the abelian groups in the homotopy Mackey functors $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ corresponding to the group $G = C_2 \times C_2$ itself, was computed recently by Holler and Kriz in [2]. We use the word additive here as the G-spectrum $H\underline{\mathbb{F}_2}$ is a commutative ring spectrum, which implies that $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is not only an $RO(G)$-graded Mackey functor but also an $RO(G)$-graded Green functor, where a Green functor can be thought of as a multiplicative analogue of a Mackey functor. In particular, we have that $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ (and moreover $\pi^H_\bigstar H\underline{\mathbb{F}_2}$ for each subgroup H of G) is a commutative ring, and a description of the ring structure does not appear in the literature.

In Chapter 3, we discuss the analogue of the above problem for the cyclic group $G = C_2$ of order two. The $RO(C_2)$-graded Green functor structure of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is known, where now $\underline{\mathbb{F}_2}$ is the constant C_2-Mackey functor associated to $\mathbb{F}_2$. For example, the Mackey functor structure of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ can be found in [3, Section 3.1] and the ring structure of $\pi^{C_2}_\bigstar H\underline{\mathbb{F}_2}$ is described in [17, Section 2] and [4, Section 6]. The portion of $\pi^{C_2}_\bigstar H\underline{\mathbb{F}_2}$ corresponding to actual representations in $RO(C_2)$, which in the literature is called the positive cone, is given by the polynomial ring $\mathbb{F}_2[x, y]$ on two particular classes x and y. The ring structure is more complicated for virtual representations, called the negative cone, whereby there exists a class θ that is divisible (in a formal sense) by monomials in $\mathbb{F}_2[x, y]$. A picture of this description of the ring structure is given in Figure 3.2.

Returning to the group $G = C_2 \times C_2$, we also call the portion of $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ corresponding to actual representations by the positive cone. However, the group $C_2 \times C_2$ has three non-trivial irreducible real representations (as opposed to one as is the case for C_2) which we denote by σ_1, σ_2 and σ_3, and so as an abelian group we have that $RO(G) \cong \mathbb{Z}\{1, \sigma_1, \sigma_2, \sigma_3\}$. We therefore break up the portion of $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ corresponding to virtual representations into the negative cone, which consists of elements in $RO(G)$ such that the coefficient of each σ_i is negative, as well as six mixed cones where there is a mix of actual and virtual representations, i.e. at least one of the coefficients of the σ_i is positive and at least one of the coefficients is negative. In Chapter 4, we give a description of the multiplicative structure of $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ and give a closed-form answer in the positive cone.

Theorem 1.1. *The positive cone in $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ is given by the quotient ring*

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3)}.$$

Here the classes x_i and y_i for $i \in \{1, 2, 3\}$ are defined in Section 4.4, and can be thought of as generating three copies of the positive cone in $\pi^{C_2}_\bigstar H\underline{\mathbb{F}_2}$ corresponding to the three C_2-subgroups of G, which we denote by H_1, H_2 and H_3. In fact, as we discuss in Section 4.6, the homology in the negative and mixed cones can be expressed in terms of the polynomial $f = x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3$. Namely, we can view the quotient ring of Theorem 1.1 as the homology of the chain complex

$$\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]\{\Xi\} \xrightarrow{f} \mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]\{1\}$$

concentrated in two degrees, where this map sends

$$m \cdot \Xi \mapsto (x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3)m \cdot 1$$

for m a monomial in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$, and this map is injective but not surjective. The negative and mixed cones can be similarly expressed as the homology of a single map given by multiplication by the polynomial f, where in the negative cone the map is surjective but not injective and in the mixed cones is neither injective nor surjective.

This algebraic description of the homology $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ can be extended to give an algebraic description of the complete Mackey functor structure $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ as we discuss in Section 4.7. Partial computations of the homotopy Mackey functors of $H\underline{\mathbb{F}_2}$ were done by Guillou and Yarnall in [3] in the context of understanding the slice spectral sequence for $\Sigma^n H\underline{\mathbb{F}_2}$ where $n \geq 0$. Namely, they computed the homotopy Mackey functors graded by multiples of the regular representation of G. The slice spectral sequence was a key tool in the Hill, Hopkins and Ravenel solution to the Kervaire invariant one problem, and the computation of all the homotopy Mackey functors in this thesis is useful when analysing the slice spectral sequence for $\Sigma^V H\underline{\mathbb{F}_2}$ given an arbitrary $V \in RO(G)$. The Mackey functor structure in the positive cone has a particularly nice form, as is the case for the ring structure of $\pi^G_\bigstar H\underline{\mathbb{F}_2}$.

Theorem 1.2. *The Mackey functor structure of the positive cone in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is given by the Mackey functor of $RO(G)$-graded rings*

$$
\begin{array}{ccccc}
 & & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_1y_2y_3+y_1x_2y_3+y_1y_2x_3)} & & \\
 & \swarrow & \downarrow & \searrow & \\
\frac{\mathbb{F}_2[y_1,x_2,y_2,x_3,y_3]}{(x_2y_3+y_2x_3)} & & \frac{\mathbb{F}_2[x_1,y_1,y_2,x_3,y_3]}{(x_1y_3+y_1x_3)} & & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(x_1y_2+y_1x_2)}, \\
 & \searrow & \downarrow & \swarrow & \\
 & & \mathbb{F}_2[y_1,y_2,y_3] & &
\end{array}
$$

where each restriction map is the identity on a generator of the domain that is also a generator of the codomain and is zero on a generator otherwise. The transfer maps are always zero.

Many of the computations in Chapter 4 can be adapted to analysing $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$ where $\underline{\mathbb{Z}}$ is the constant Mackey functor associated to the integers $\mathbb{Z}$. Indeed, we construct an explicit chain complex of Mackey functors in Section 4.2 whose homology at each tridegree gives us $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$, and an analogous chain complex can be constructed with integer coefficients. We discuss this briefly in Section 4.8 in the context of the Bockstein spectral sequence and how it can be used to deduce $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$ from $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$, at least in the positive cone. The result with integer coefficients is not in the literature, however it can be found for the group C_2 in [22, Theorem 2.8]. The collapsing of the Bockstein spectral sequence to the E^2-page that we prove in Section 4.8 has been used by McCall [15] in understanding the slice spectral sequence for C_2-equivariant Real K-theory.

The structure of the thesis is as follows. In Chapter 2, we discuss the necessary background in equivariant stable homotopy theory needed in later chapters, including the definition of G-spectra, the notion of Mackey functors and Green functors as well as the concept of the homotopy Mackey functors of a G-spectrum. In Chapter 3, we compute the known $RO(C_2)$-graded Green functor structure of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ and discuss evident symmetries in the result which can be explained using Anderson duality. The heart of the thesis is Chapter 4, where we generalise the results and methods of Chapter 3 to the Klein four group.

Chapter 2

Background

In this chapter, we introduce some basic concepts of equivariant stable homotopy theory that will be assumed in later chapters. Furthermore, we will be working with an arbitrary finite group G throughout this chapter, even though we will be focusing on the cyclic group C_2 of order two and the Klein four-group $C_2 \times C_2$ in Chapters 3 and 4. See [24] and [19] for a thorough introduction to equivariant stable homotopy theory.

2.1 The category of orthogonal G-spectra

Let G be an arbitrary finite group, which we fix throughout this chapter. Furthermore, when we use the word *space* we mean a compactly generated weak Hausdorff topological space. Our goal in this section is to introduce the fundamental objects in equivariant stable homotopy theory, namely orthogonal G-spectra. The first step however is to understand the notion of G-spaces.

Definition 2.1. A *G-space* is a space X together with an action of the group G. A *G-equivariant map* (or simply *equivariant map*) $f\colon X \to Y$ of G-spaces is a map of the underlying spaces (i.e. a continuous map) that commutes with the G-actions on X and Y. That is, we require that the diagram

$$\begin{array}{ccc} X & \xrightarrow{g\cdot} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g\cdot} & Y \end{array}$$

commutes for every $g \in G$. A *pointed G-space* is a G-space equipped with a basepoint fixed by the action of G.

Any G-space X can be made into a pointed G-space by adding in a disjoint basepoint which we define to be fixed by the G-action, and we denote the resulting pointed G-space by X_+. Now, we define the category $\mathcal{T}op^G$ to have G-spaces as its objects and equivariant maps as its morphisms. Similarly, we define the category $\mathcal{T}^G$ to have pointed G-spaces as its objects and equivariant maps (which preserve the basepoints) as its morphisms. However, we define the category $\mathcal{T}_G$ to have pointed G-spaces as its objects but continuous maps as its morphisms, i.e. we allow for non-equivariant maps between G-spaces in the category $\mathcal{T}_G$. Both $\mathcal{T}^G$ and $\mathcal{T}_G$ are closed symmetric monoidal categories under smash product with the 0-sphere S^0 as the unit (with trivial G-action). Note here that the G-action on the smash product of pointed G-spaces is induced by the diagonal G-action.

Furthermore, notice that the category $\mathcal{T}_G$ is enriched over $\mathcal{T}^G$. Indeed, each morphism object $\mathcal{T}_G(X, Y)$ in $\mathcal{T}_G$ is the space of all continuous maps $f\colon X \to Y$ and we can equip this space with the conjugate G-action $(g \cdot f)(x) = gf(g^{-1}x)$, so we can identify $\mathcal{T}_G(X, Y)$ as an object of $\mathcal{T}^G$. Furthermore, a simple calculation shows that composition

$$\mathcal{T}_G(Y, Z) \wedge \mathcal{T}_G(X, Y) \to \mathcal{T}_G(X, Z)$$

in $\mathcal{T}_G$ is equivariant with respect to the conjugate G-action. We have that $\mathcal{T}^G$ is a subcategory of $\mathcal{T}_G$, and thus the category $\mathcal{T}_G$ is enriched over itself. The category $\mathcal{T}^G$ is enriched over the category $\mathcal{T}$ of pointed spaces with continuous maps preserving the basepoints, and the morphism objects $\mathcal{T}^G(X, Y)$ in the category $\mathcal{T}^G$ are precisely the G-fixed points $\mathcal{T}_G(X, Y)^G$ of the morphism objects in $\mathcal{T}_G$ following from the definition of the conjugate G-action.

In order to define orthogonal G-spectra, we need to understand the *Mandell-May category* $\mathcal{J}_G$. First, recall that a finite-dimensional real representation V of G is called *orthogonal* if the underlying finite-dimensional real vector space V is also an inner product space, and each $g \in G$ acts on V as an element of the group $O(V)$ of orthogonal endomorphisms of V, i.e. endomorphisms of V that preserve the inner product. Now, given finite-dimensional real orthogonal G-representations V and W, let $O(V, W)$ denote the space of orthogonal (not necessarily equivariant) embeddings $V \to W$. Note that G acts on $O(V, W)$ via the conjugate action. Then, consider the space

$$\{(f, w) \in O(V, W) \times W : w \in f(V)^{\perp} \subset W\},$$

which is in fact a vector bundle on $O(V, W)$ under the projection map onto the first component.

Definition 2.2. The objects of the *Mandell-May category* $\mathscr{J}_G$ are finite-dimensional real orthogonal G-representations V. Given two objects V and W in the category $\mathscr{J}_G$, we define the morphism set $\mathscr{J}_G(V,W)$ to be the Thom space of the above vector bundle on $O(V,W)$.

Recall that the Thom space of a vector bundle is the quotient space obtained by taking the one-point compactification of each of the fibres and identifying all of the points-at-infinity. That is, as a set we have that

$$\mathscr{J}_G(V,W) = \bigvee_{f \in O(V,W)} S^{f(V)^\perp}.$$

The Mandell-May category $\mathscr{J}_G$ is enriched over $\mathcal{T}^G$. Indeed, each $\mathscr{J}_G(V,W)$ is a pointed G-space with basepoint the point-at-infinity and has G-action induced by the action of G on each orthogonal complement $f(V)^\perp \subset W$ for $f \in O(V,W)$. Furthermore, the map

$$\mathscr{J}_G(V,W) \wedge \mathscr{J}_G(U,V) \to \mathscr{J}_G(U,W)$$

induced by composition of orthogonal embeddings is G-equivariant.

Example 2.3. If $V = 0$, then the Thom space $\mathscr{J}_G(V,W)$ is the representation sphere S^W, i.e. the one-point compactification of W with G acting trivially on the point-at-infinity, which follows since there is only one embedding $0 \to W$ and the orthogonal complement of the image of this embedding is W.

Example 2.4. If the dimension of V is greater than the dimension of W as finite-dimensional real vector spaces, then $O(V,W)$ is empty so the Thom space $\mathscr{J}_G(V,W)$ is a point, i.e. only consists of the point-at-infinity. If the dimensions of V and W are equal, then the embedding space is $O(V)$ and the orthogonal complement of any element of $O(V)$ is zero. Hence, the Thom space in this case is $O(V)_+$ with G acting on $O(V)$ via the conjugate action.

Now that we understand the Mandell-May category $\mathscr{J}_G$, we are ready to define orthogonal G-spectra.

Definition 2.5. The category $\mathcal{S}p^G$ of *orthogonal G-spectra* is the enriched functor category $[\mathscr{J}_G, \mathcal{T}^G]$. That is, an *orthogonal G-spectrum* X is an enriched functor $\mathscr{J}_G \to \mathcal{T}^G$.

By Definition 2.5, we can therefore think of an orthogonal G-spectrum X as consisting of a pointed G-space X_V for each finite-dimensional real orthogonal G-representation V, together with structure maps

$$\mathscr{J}_G(V, W) \wedge X_V \to X_W$$

which we require to be equivariant, i.e. that the structure maps are morphisms in the category $\mathcal{T}^G$. Note that the morphisms in the category $\mathcal{S}p^G$ are natural transformations of enriched functors $\mathscr{J}_G \to \mathcal{T}^G$. We will often drop the word 'orthogonal' and refer to objects of $\mathcal{S}p^G$ simply as G-spectra. Furthermore, morphisms in the category $\mathcal{S}p^G$ will be called *equivariant maps* of G-spectra.

Definition 2.6. If K is a pointed G-space, then its *suspension G-spectrum* $\Sigma^\infty K$ is defined by

$$(\Sigma^\infty K)_V = \Sigma^V K$$

for all objects V in $\mathscr{J}_G$, where $\Sigma^V K := S^V \wedge K$.

We can also more generally take the smash product of a pointed G-space with a G-spectrum in the sense of Definition 2.7.

Definition 2.7. Let K be a pointed G-space and X a G-spectrum. Then, their *smash product* is the G-spectrum $K \wedge X$ defined by

$$(K \wedge X)_V = K \wedge X_V$$

for all objects V in $\mathscr{J}_G$.

Note that Definitions 2.6 and 2.7 are using that the category $\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G]$ of G-spectra is tensored over $\mathcal{T}^G$, so an equivariant map $f\colon K \to L$ of pointed G-spaces induces a map

$$f \wedge X\colon K \wedge X \to L \wedge X$$

of G-spectra for every G-spectrum X. We define the smash product of G-spectra as follows.

Definition 2.8. Let X and Y be objects in $\mathcal{S}p^G = [\mathscr{J}_G, \mathcal{T}^G]$. Then, their *smash product* $X \wedge Y$ is defined to be the left Kan extension of the composite $\wedge \circ (X \times Y)$ of the product of the functors X and Y with the smash product on pointed G-spaces along the direct sum $\oplus$ of finite-dimensional real orthogonal representations:

$$\begin{array}{ccccc} \mathscr{J}_G \times \mathscr{J}_G & \xrightarrow{X\times Y} & \mathcal{T}^G \times \mathcal{T}^G & \xrightarrow{\wedge} & \mathcal{T}^G \\ & \searrow^{\oplus} & & \nearrow_{X\wedge Y} & \\ & & \mathscr{J}_G & & \end{array}$$

That is, the functor $X \wedge Y$ is initial among functors $Z\colon \mathscr{J}_G \to \mathcal{T}^G$ with natural transformations

$$\wedge \circ (X \times Y) \implies Z \circ \oplus.$$

This definition is an example of Day convolution as given in [5].

Theorem 2.9. *The category $\mathcal{S}p^G$ of orthogonal G-spectra is a closed symmetric monoidal category with respect to the smash product of Definition 2.8. The unit for the smash product is the sphere spectrum S^{-0} defined by $(S^{-0})_V = \mathscr{J}_G(0,V) = S^V$.*

Proof. This follows from the Day convolution theorem, using that $\mathscr{J}_G$ is a small symmetric monoidal category (with respect to direct sum) enriched over $\mathcal{T}^G$, and $\mathcal{T}^G$ is a cocomplete closed symmetric monoidal category with respect to the smash product of pointed G-spaces. □

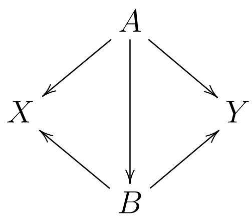
Remark 2.10. The notation S^{-0} for the sphere spectrum as in the statement of Theorem 2.9 was adopted by Hill, Hopkins and Ravenel to help distinguish it from the ordinary sphere S^0, which has previously also been used to denote the sphere spectrum. More generally for an actual G-representation U we define the G-spectrum S^{-U} by $(S^{-U})_V = \mathscr{J}_G(U,V)$ known as a *virtual representation sphere* as we will see in Section 2.3.

2.2 Mackey and Green functors

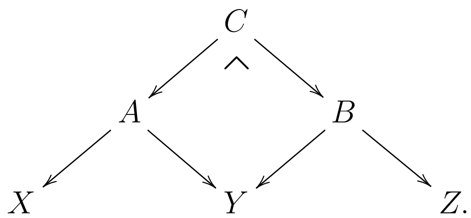
In this section we introduce the concepts of Mackey functors and Green functors, which will play a central role throughout later chapters. One may think of Mackey functors as generalising the role that abelian groups play in non-equivariant stable homotopy theory. For example, when computing singular (co)homology of a space we take coefficients in some abelian group, whereas when we consider equivariant (co)homology theories our coefficient abelian group becomes a coefficient Mackey functor. Informally, a G-Mackey functor consists of an abelian group for each subgroup of G along with transfer and restriction maps between these abelian groups.

We will give one of two equivalent definitions of a Mackey functor. Although the definition that we will not present more closely resembles the above informal notion of a Mackey functor, the definition that we do give will be more useful when we define Green functors and the box product of Mackey functors, and is also more elegant in that it defines a Mackey functor as indeed a single functor. First, we need to define the *Lindner category* $\mathscr{B}_G^+$. Let $\mathcal{F}_G$ denote the category whose objects are finite G-sets and whose morphisms are equivariant maps.

Definition 2.11. The objects of the *Lindner category* $\mathscr{B}_G^+$ are precisely the objects of $\mathcal{F}_G$, i.e. finite G-sets. However, a morphism from a finite G-set X to a finite G-set Y in the category $\mathscr{B}_G^+$ is an equivalence class of diagrams (which we call *spans*) of the form $X \leftarrow A \rightarrow Y$, where the maps $A \rightarrow X$ and $A \rightarrow Y$ are morphisms in $\mathcal{F}_G$. Two such spans $X \leftarrow A \rightarrow Y$ and $X \leftarrow B \rightarrow Y$ are defined to be equivalent if there is an isomorphism $A \rightarrow B$ in $\mathcal{F}_G$ (i.e. a bijective equivariant map) such that the diagram



commutes. The composite of two morphisms represented by the spans $X \leftarrow A \rightarrow Y$ and $Y \leftarrow B \rightarrow Z$ is the equivalence class of the span $X \leftarrow C \rightarrow Z$ obtained from the pullback diagram



For all objects X and Y in $\mathscr{B}_G^+$, the morphism set $\mathscr{B}_G^+(X, Y)$ is an abelian monoid under disjoint union, where the disjoint union of the morphisms represented by $X \leftarrow A \rightarrow Y$ and $X \leftarrow B \rightarrow Y$ is the equivalence class of the span $X \leftarrow A \sqcup B \rightarrow Y$ with the evident maps $A \sqcup B \rightarrow X$ and $A \sqcup B \rightarrow Y$. Note that the zero object of the abelian monoid $\mathscr{B}_G^+(X, Y)$ is the equivalence class of the span $X \leftarrow \emptyset \rightarrow Y$.

Definition 2.12. The objects of the *Burnside category* $\mathscr{B}_G$ are precisely the objects of $\mathcal{F}_G$ (and of $\mathscr{B}_G^+$). However, each morphism set $\mathscr{B}_G(X, Y)$ is the group completion of the abelian monoid $\mathscr{B}_G^+(X, Y)$.

In particular, the Burnside category $\mathscr{B}_G$ is enriched over the category Ab of abelian groups. Now that we have constructed the Burnside category, we are ready to define a Mackey functor.

Definition 2.13. A *Mackey functor* (or *G-Mackey functor*) is an additive functor $\underline{M}\colon \mathscr{B}_G \to Ab$. The functor $\underline{M}$ being additive means that it is an enriched functor over Ab and it sends disjoint unions of finite G-sets to direct sums of abelian groups.

An equivalent definition of a Mackey functor as mentioned earlier is given in [1, Definition 3.1]. The category Mack(G) of G-Mackey functors is an abelian category with addition defined in terms of the addition on each level, i.e. on the image $\underline{M}(X)$ of each finite G-set X. Note that we could have equivalently defined a Mackey functor as a contravariant functor $\underline{M}\colon \mathscr{B}_G^{op} \to Ab$ since following from how we defined morphisms in $\mathscr{B}_G$ in terms of equivalence classes of spans we have that $\mathscr{B}_G^{op} \cong \mathscr{B}_G$. We will always use an underline to denote a Mackey functor as in Definition 2.13.

Since any finite G-set can be decomposed as a disjoint union of orbits of the form G/H for H a subgroup of G and a Mackey functor sends disjoint unions to direct sums, it suffices to know how a Mackey functor behaves on the full subcategory $\mathscr{O}_G$ of $\mathcal{F}_G$ where $\mathscr{O}_G$ is the orbit category of G with orbits G/H as its objects.

Definition 2.14. Let $\underline{M}$ be a Mackey functor and let K and H be subgroups of G with $K \subset H \subset G$. Then, the *restriction map* $\mathrm{Res}_K^H\colon \underline{M}(G/H) \to \underline{M}(G/K)$ is the image under $\underline{M}$ of the morphism represented by the span

$$G/H \leftarrow G/K \xrightarrow{id} G/K,$$

where $G/K \to G/H$ is the projection map. The *transfer map* $\mathrm{Tr}_K^H\colon \underline{M}(G/K) \to \underline{M}(G/H)$ is the image under $\underline{M}$ of the morphism represented by the span

$$G/K \xleftarrow{id} G/K \to G/H.$$

The Weyl group $W_G(H)$ acts on $\underline{M}(G/H)$ as follows, where we think of $W_G(H)$ as the group of isomorphisms from G/H to itself in the category $\mathscr{O}_G$.

Given an element $\gamma\colon G/H \to G/H$ in the Weyl group $W_G(H)$ and an element $x \in \underline{M}(G/H)$, we define $\gamma \cdot x \in \underline{M}(G/H)$ to be the image of x under the morphism $\underline{M}(G/H) \to \underline{M}(G/H)$ obtained by applying $\underline{M}$ to the morphism in $\mathscr{B}_G$ represented by the span

$$G/H \xleftarrow{\gamma} G/H \xrightarrow{id} G/H.$$

The Weyl group action on the levels of a Mackey functor is particularly useful as we have a formula for the restriction of an element in the image of a transfer in a Mackey functor in terms of this group action.

Proposition 2.15. *Let H be a subgroup of G and let K be a normal subgroup of H. Then, for each $x \in \underline{M}(G/K)$ we have that*

$$Res^H_K(Tr^H_K(x)) = \sum_{\gamma \in W_K(H)} \gamma \cdot x.$$

We now give two important examples of G-Mackey functors that will be used extensively in later chapters.

Example 2.16. Let M be an abelian group. The *constant Mackey functor* $\underline{M}$ associated to M is given by $\underline{M}(G/H) = M$ for all subgroups H of G. If K is a subgroup of H, then the restriction map $\mathrm{Res}^H_K\colon M \to M$ is the identity map and the transfer map $\mathrm{Tr}^H_K\colon M \to M$ is multiplication by the index $[H : K]$.

Example 2.17. Let M be a G-module. The *fixed point Mackey functor* $\underline{M}$ associated to M is given by $\underline{M}(G/H) = M^H$ for all subgroups H of G, where M^H is the (abelian) group of elements of M fixed by the action of H. If K is a subgroup of H, then the restriction map $\mathrm{Res}^H_K\colon M^H \to M^K$ is the inclusion of fixed points, noting that indeed if $x \in M$ is fixed by H then it is in particular fixed by $K \subset H$. The transfer map $\mathrm{Tr}^H_K\colon M^K \to M^H$ is given by

$$\mathrm{Tr}^H_K(x) = \sum_{\bar{h} \in H/K} h \cdot x,$$

which is independent of choice of representatives for cosets in H/K.

Notice that the constant Mackey functor associated to an abelian group is a special case of a fixed point Mackey functor. Indeed, if M is an abelian group then we can treat M as a G-module with trivial G-action. Then, the fixed point Mackey functor associated to the G-module M is precisely the constant Mackey functor associated to the abelian group M. Another important example is the Burnside Mackey functor.

Example 2.18. The *Burnside Mackey functor* $\underline{A}_G$ is defined by $\underline{A}_G(G/H) = \mathscr{B}_G(G/G, G/H)$ for all subgroups H of G. Since we can ignore the left-hand map in a span $G/G \leftarrow X \rightarrow G/H$, we can identify $\mathscr{B}_G(G/G, G/H)$ as the group completion of the abelian monoid of isomorphism classes of finite G-sets over G/H, i.e. equivariant maps from finite G-sets to G/H, under disjoint union. If K and H are subgroups of G with $K \subset H$, then the transfer map $\mathrm{Tr}_K^H \colon \mathscr{B}_G(G/G, G/K) \rightarrow \mathscr{B}_G(G/G, G/H)$ is given by composing maps $X \rightarrow G/K$ with the projection $G/K \rightarrow G/H$, and the restriction map $\mathrm{Res}_K^H \colon \mathscr{B}_G(G/G, G/H) \rightarrow \mathscr{B}_G(G/G, G/K)$ is given by taking the pullback of a map $X \rightarrow G/H$ along the projection $G/K \rightarrow G/H$.

Remark 2.19. Since the category of finite G-sets over G/H is equivalent to the category of finite H-sets, we have that $\underline{A}_G(G/H) = A(H)$ where $A(H)$ is the *Burnside ring* of H, i.e. the group completion of the abelian monoid of isomorphism classes of finite H-sets under disjoint union with multiplication induced by the product of finite H-sets.

We also have the notion of the inflation of a Mackey functor along a quotient map, following the notation of [3, Definition 2.3], as well as the notion of the restriction of a Mackey functor to a subgroup.

Definition 2.20. If N is a normal subgroup of G, then the quotient map $\phi_N \colon G \rightarrow G/N$ induces a functor $\phi_N^* \colon \mathrm{Mack}(G/N) \rightarrow \mathrm{Mack}(G)$ where if $\underline{M}$ is a G/N-Mackey functor then the G-Mackey functor $\phi_N^*(\underline{M})$ is defined by

$$\phi_N^*(\underline{M})(G/H) = \begin{cases} \underline{M}(H/N) & \text{if } N \subset H, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.21. Let $\underline{M}$ be a G-Mackey functor and H a subgroup of G. Then, the *restricted H-Mackey functor* $\downarrow_H^G \underline{M}$ is defined by

$$(\downarrow_H^G \underline{M})(T) = \underline{M}(G \times_H T)$$

for all finite H-sets T, where the finite G-set $G \times_H T$ is the quotient of $G \times T$ by the relation $(gh, t) \sim (g, ht)$ for all $h \in H$, with G action given by left-multiplication in the first component.

Although Proposition 2.15 gives us a formula for the restriction of a transfer in a Mackey functor, we do not in general have a formula for the transfer of a restriction.

Definition 2.22. A Mackey functor $\underline{M}$ is called *cohomological* if for all subgroups K and H of G with $K \subset H$, we have that

$$\mathrm{Tr}_K^H(\mathrm{Res}_K^H(x)) = [H : K]x$$

for every $x \in \underline{M}(G/H)$.

As given by the following proposition, we have already seen examples of cohomological Mackey functors.

Proposition 2.23. *Fixed point Mackey functors are cohomological.*

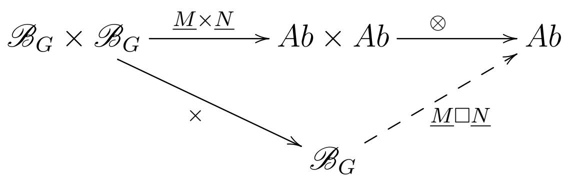
Proof. Let M be a G-module, and let K and H be subgroups of G with $K \subset H$. Given an arbitrary element $x \in M^H$, by definition we have that $\mathrm{Res}_K^H(x) = x$. However, we also have by definition that

$$\mathrm{Tr}_K^H(x) = \sum_{\bar{h} \in H/K} h \cdot x = [H : K]x$$

as H acts trivially on x. □

Our next goal is to define Green functors, which can be thought of as multiplicative analogues of Mackey functors, and to do this we first define the box product of Mackey functors. Since we have defined the category Mack(G) of G-Mackey functors as the functor category $[\mathscr{B}_G, Ab]$ enriched over Ab, we can again use the Day convolution to put a symmetric monoidal structure on Mack(G).

Definition 2.24. The *box product* $\underline{M} \Box \underline{N}$ of two Mackey functors $\underline{M}$ and $\underline{N}$ is defined to be the left Kan extension of the composite $\otimes \circ (\underline{M} \times \underline{N})$ of the product of the functors $\underline{M}$ and $\underline{N}$ with the tensor product of abelian groups, along the direct product $\times$ of finite G-sets:



Similar to Theorem 2.9, the Day convolution theorem implies that Mack(G) under the box product is a closed symmetric monoidal category. The unit for the box product is the Burnside Mackey functor $\underline{A}$ of Example 2.18. In particular, we notice that a map (i.e. a natural transformation) $\underline{M} \Box \underline{N} \to \underline{P}$ of Mackey functors is equivalent to maps

$$\underline{M}(X) \otimes \underline{N}(Y) \to \underline{P}(X \times Y)$$

which are natural in X and Y.

If we let cMack(G) denote the full subcategory of cohomological Mackey functors, then cMack(G) is still a closed symmetric monoidal category as the box product of two cohomological Mackey functors is cohomological. However, the Burnside Mackey functor $\underline{A}$ is not cohomological so is not the unit in cMack(G), and in fact the unit is the constant Mackey functor $\underline{\mathbb{Z}}$ (see [13, Proposition 2.3.5]).

Definition 2.25. A *Green functor* is an abelian monoid in Mack(G) under the box product $\square$ of Mackey functors.

We will also give an equivalent yet more explicit definition of a Green functor that will be useful in later chapters, and for this we need to understand the box product from a more explicit viewpoint.

Definition 2.26. If $\underline{M}$, $\underline{N}$ and $\underline{P}$ are Mackey functors, then a *Dress pairing* $\varphi\colon (\underline{M}, \underline{N}) \to \underline{P}$ consists of a collection of maps

$$\varphi_H \colon \underline{M}(G/H) \otimes \underline{N}(G/H) \to \underline{P}(G/H)$$

for each subgroup H of G, which satisfy the following properties. The maps φ_H are required to commute with the restriction maps so that the diagram

$$\begin{array}{ccc} \underline{M}(G/H) \otimes \underline{N}(G/H) & \xrightarrow{\varphi_H} & \underline{P}(G/H) \\ {\scriptstyle \mathrm{Res}^H_K \otimes \mathrm{Res}^H_K}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^H_K} \\ \underline{M}(G/K) \otimes \underline{N}(G/K) & \xrightarrow{\varphi_K} & \underline{P}(G/K) \end{array}$$

commutes. Furthermore, we require that the diagrams

$$\begin{array}{cccc} & \underline{M}(G/K) \otimes \underline{N}(G/K) & \xrightarrow{\varphi_K} & \underline{P}(G/K) \\ & {\scriptstyle \mathrm{Res}^H_K \otimes id}\nearrow & & \\ \underline{M}(G/H) \otimes \underline{N}(G/K) & & & \downarrow{\scriptstyle \mathrm{Tr}^H_K} \\ & {\scriptstyle id \otimes \mathrm{Tr}^H_K}\searrow & & \\ & \underline{M}(G/H) \otimes \underline{N}(G/H) & \xrightarrow{\varphi_H} & \underline{P}(G/H) \end{array}$$

and

$$\begin{array}{cccc} & \underline{M}(G/K) \otimes \underline{N}(G/K) & \xrightarrow{\varphi_K} & \underline{P}(G/K) \\ & {\scriptstyle id \otimes \mathrm{Res}^H_K}\nearrow & & \\ \underline{M}(G/K) \otimes \underline{N}(G/H) & & & \downarrow{\scriptstyle \mathrm{Tr}^H_K} \\ & {\scriptstyle \mathrm{Tr}^H_K \otimes id}\searrow & & \\ & \underline{M}(G/H) \otimes \underline{N}(G/H) & \xrightarrow{\varphi_H} & \underline{P}(G/H) \end{array}$$

commute. Element-wise, the commutativity of these diagrams means that

$$\varphi_H(a \otimes \mathrm{Tr}_K^H(b)) = \mathrm{Tr}_K^H(\varphi_K(\mathrm{Res}_K^H(a) \otimes b))$$

and

$$\varphi_H(\mathrm{Tr}_K^H(c) \otimes d) = \mathrm{Tr}_K^H(\varphi_K(c \otimes \mathrm{Res}_K^H(d))),$$

which is called *Frobenius reciprocity.*

The reason why we are interested in Dress pairings is that there is a close relationship between maps out of the box product of Mackey functors and Dress pairings. The following lemma is stated without proof in [12] and we sketch the proof given in [11, Lemma 2.17].

Lemma 2.27. *Given Mackey functors $\underline{M}$, $\underline{N}$ and $\underline{P}$, there is a one-to-one correspondence between maps $\psi\colon \underline{M}\Box\underline{N} \to \underline{P}$ out of the box product of $\underline{M}$ and $\underline{N}$ and Dress pairings $\varphi\colon (\underline{M}, \underline{N}) \to \underline{P}$.*

Proof Sketch. Suppose that we start with a map $\psi\colon \underline{M}\Box\underline{N} \to \underline{P}$. As discussed earlier, this is equivalent to having maps $\underline{M}(X) \otimes \underline{N}(Y) \to \underline{P}(X \times Y)$ that are natural in the finite G-sets X and Y. We define a Dress pairing $\varphi\colon (\underline{M}, \underline{N}) \to \underline{P}$ as follows. For each subgroup H of G, the above gives us a map

$$\psi_H\colon \underline{M}(G/H) \otimes \underline{N}(G/H) \to \underline{P}(G/H \times G/H).$$

However, we also have a map $f\colon \underline{P}(G/H \times G/H) \to \underline{P}(G/H)$ defined by the image under $\underline{P}$ of the morphism in $\mathscr{B}_G$ given by the equivalence class of the span $G/H \times G/H \xleftarrow{\Delta} G/H \xrightarrow{id} G/H$. Then, the collection of all $\varphi_H := f \circ \psi_H$ defines a Dress pairing.

For the other direction, suppose that we start with a Dress pairing $\varphi\colon (\underline{M}, \underline{N}) \to \underline{P}$, and let X and Y be arbitrary finite G-sets. Consider the maps $f\colon \underline{M}(X) \to \underline{M}(X \times Y)$ and $g\colon \underline{N}(Y) \to \underline{N}(X \times Y)$ defined as the image under $\underline{M}$ and $\underline{N}$ of the morphisms in $\mathscr{B}_G$ represented by the spans $X \xleftarrow{pr_1} X \times Y \xrightarrow{id} X \times Y$ and $Y \xleftarrow{pr_2} X \times Y \xrightarrow{id} X \times Y$ respectively. Taking their tensor product gives us a map

$$f \otimes g\colon \underline{M}(X) \otimes \underline{N}(Y) \to \underline{M}(X \times Y) \otimes \underline{N}(X \times Y).$$

Now, we know that any finite G-set can be decomposed into a disjoint union of orbits, so write

$$X \times Y = \coprod_{\alpha} G/H_\alpha.$$

Since Mackey functors take disjoint unions of finite G-sets to direct sums of abelian groups, it follows that

$$\underline{M}(X \times Y) \otimes \underline{N}(X \times Y) = C \oplus \bigoplus_{\alpha} (\underline{M}(G/H_\alpha) \otimes \underline{N}(G/H_\alpha)),$$

where C consists of the crossed-terms obtained when we distributed the tensor product over the direct sum. Thus, since the Dress pairing φ gives us maps $\varphi_{H_\alpha} \colon \underline{M}(G/H_\alpha) \otimes \underline{N}(G/H_\alpha) \to \underline{P}(G/H_\alpha)$ for all α, we have a map

$$C \oplus \bigoplus_{\alpha} (\underline{M}(G/H_\alpha) \otimes \underline{N}(G/H_\alpha)) \xrightarrow{0 \oplus \bigoplus \varphi_{H_\alpha}} \bigoplus_{\alpha} \underline{P}(G/H_\alpha) = \underline{P}(X \times Y).$$

By composing $f \otimes g$ with this map, we obtain a map $\underline{M}(X) \otimes \underline{N}(Y) \to \underline{P}(X \times Y)$ that is natural in X and Y. □

Using Lemma 2.27, it then follows that the box product of Mackey functors has the following explicit inductively defined formula.

Proposition 2.28. *Let $\underline{M}$ and $\underline{N}$ be Mackey functors. If H is a subgroup of G, then $(\underline{M} \square \underline{N})(G/H)$ is given in terms of $(\underline{M} \square \underline{N})(G/K)$ for each subgroup K of H as*

$$(\underline{M} \square \underline{N})(G/H) = \underline{M}(G/H) \otimes \underline{N}(G/H) \oplus \bigoplus_{K<H} ((\underline{M} \square \underline{N})(G/K)/W_K(H))/FR,$$

with the evident transfer and restriction maps. Here FR is the Frobenius reciprocity submodule and is generated by elements of the form

$$a \otimes Tr_K^H(b) - Tr_K^H(Res_K^H(a) \otimes b)$$

and

$$Tr_K^H(c) \otimes d - Tr_K^H(c \otimes Res_K^H(d)).$$

Explicit computations of the box product when $G = C_p$ can be found in [9], and see [10] and [12] for further discussion on the box product of Mackey functors. As in Definition 2.25, we can now view a Green functor as an abelian monoid under this explicit description of the box product given by Proposition 2.28. One can show that we now have the following equivalent definition of a Green functor that we will primarily use in later chapters.

Definition 2.29. A *Green functor* is a Mackey functor $\underline{R}$ such that $\underline{R}(G/H)$ is a commutative ring for each subgroup H of G. Furthermore, we require that each restriction map $\mathrm{Res}^H_K\colon \underline{R}(G/H) \to \underline{R}(G/K)$ is a ring map, and that the transfer maps satisfy Frobenius reciprocity, i.e. that

$$\mathrm{Tr}^H_K(x) \cdot y = \mathrm{Tr}^H_K(x \cdot \mathrm{Res}^H_K(y))$$

for all $x \in \underline{R}(G/K)$ and $y \in \underline{R}(G/H)$.

We will be primarily be working with Mackey and Green functors in the context of the homotopy of G-spectra.

2.3 The homotopy of equivariant spectra

In this section we discuss how Mackey functors and Green functors are used in equivariant stable homotopy theory. Non-equivariantly, we have the notion of the homotopy groups $\pi_n(X)$ of a spectrum X where $n \in \mathbb{Z}$. If $n \geq 0$, we define

$$\pi_n(X) = [\Sigma^\infty S^n, X],$$

i.e. the set of homotopy classes of maps from the suspension spectrum of the n-sphere S^n to X. Note here that a non-equivariant spectrum is defined as in Section 2.1 where we take G to be the trivial group e. If $n < 0$, we define

$$\pi_n(X) = [S^n, X]$$

where the spectrum S^n is defined by $(S^n)_k = \mathscr{J}_e(-n, k)$ and $\mathscr{J}_e$ is the Mandell-May category of Definition 2.2 associated to the trivial group. The direct sum

$$\pi_*(X) = \bigoplus_{n \in \mathbb{Z}} \pi_n(X)$$

of all the homotopy groups of the spectrum X is precisely the homology of a point with respect to the generalised homology theory on the category of spectra corresponding to X. If X is a ring spectrum, i.e. a spectrum together with an associative and unital (up to homotopy) multiplication map $\mu\colon X \wedge X \to X$, then $\pi_*(X)$ is a $\mathbb{Z}$-graded ring and is called the *coefficient ring* of X.

If we look at G-spectra for an arbitrary finite group G, then we get a collection of homotopy G-Mackey functors. Furthermore, we can grade this collection of homotopy Mackey functors over the orthogonal representation ring $RO(G)$ rather than just over $\mathbb{Z}$ as in the case of non-equivariant spectra.

Definition 2.30. The *orthogonal representation ring* $RO(G)$ is the group completion of the abelian monoid of isomorphism classes of finite-dimensional real orthogonal G-representations under direct sum. The multiplication on $RO(G)$ is induced by the tensor product of orthogonal G-representations.

We will see examples of the ring $RO(G)$ for the groups $G = C_2$ and $G = C_2 \times C_2$ in Chapters 3 and 4 respectively. We now introduce the $RO(G)$-graded homotopy Mackey functors $\underline{\pi}_\bigstar(X)$ of a G-spectrum X, noting that we use the symbol $\bigstar$ for $RO(G)$-grading, the symbol $*$ for integer grading and as usual an underline for Mackey functors.

Definition 2.31. Let X be a G-spectrum. If V is a finite-dimensional real orthogonal G-representation, then the V^{th} *homotopy Mackey functor* of X is given by

$$\underline{\pi}_V(X)(T) = [S^V \wedge \Sigma^\infty T_+, X]^G$$

for all finite G-sets T, i.e. objects of the Burnside category $\mathscr{B}_G$. Here $\Sigma^\infty T_+$ is the suspension spectrum of the pointed G-set T_+ and S^V is the representation sphere (or one-point compactification) of V, and we are looking at the group of equivariant maps from the G-spectrum $S^V \wedge \Sigma^\infty T_+$ to the G-spectrum X. If V is an n-dimensional trivial representation, then we denote the corresponding homotopy Mackey functor by $\underline{\pi}_n(X)$.

If $[U] - [W]$ is a virtual G-representation giving an element of $RO(G)$, where U and W are actual orthogonal G-representations, then we have a G-spectrum S^{U-W} (which we call a *virtual representation sphere*) defined by

$$(S^{U-W})_V = \mathscr{J}_G(W, U \oplus V)$$

for all objects V in the Mandell-May category $\mathscr{J}_G$.

Remark 2.32. It is a priori not clear whether the notion of the virtual representation sphere is independent of choice of representatives for elements in $RO(G)$. However, this issue is resolved in [23, Theorem 1.6] so that indeed if $[U] - [W] = [U'] - [W']$ in $RO(G)$, then we have an equivalence $S^{U-W} \simeq S^{U'-W'}$ up to a canonical choice.

Using the notion of the virtual representation sphere, we can extend Definition 2.31 to virtual representations $V = U - W \in RO(G)$ by defining

$$\underline{\pi}_V(X)(T) = [S^V \wedge \Sigma^\infty T_+, X]^G,$$

where now $S^V \wedge \Sigma^\infty T_+$ is the smash product of two G-spectra as in Definition 2.8. As discussed in Section 2.2, it suffices to consider $\underline{\pi}_V(X)$ on the orbits G/H for H a subgroup of G, and for ease of notation we will write

$$\underline{\pi}_V(X)(G/H) = \pi_V^H(X),$$

which in this case is given by the H-fixed points $[S^V, X]^H$. Given a subgroup $K \subset H$, we have transfer and restriction maps $\mathrm{Tr}_K^H \colon \pi_V^K(X) \to \pi_V^H(X)$ and $\mathrm{Res}_K^H \colon \pi_V^H(X) \to \pi_V^K(X)$ induced by the projection map $G/K \to G/H$ as in Definition 2.14. Analogous to the fact mentioned earlier that if X is an ordinary homotopy commutative ring spectrum then $\pi_*(X)$ is a commutative ring, we have the following.

Theorem 2.33. *If X is a homotopy commutative G-ring spectrum, then $\underline{\pi}_\bigstar(X)$ is an $RO(G)$-graded Green functor.*

In particular, we have that the multiplication from the ring spectrum X induces the box product of Mackey functors. Now, recall that non-equivariantly for any abelian group A there is a corresponding Eilenberg-MacLane spectrum HA with the defining property that its homotopy groups are given by

$$\pi_n(HA) = \begin{cases} A & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

If A is a ring, then HA is a ring spectrum. Furthermore, the spectrum HA represents (in the sense of the Brown representability theorem) singular homology and singular cohomology with coefficients in the abelian group A. We similarly have an equivariant Eilenberg-MacLane spectrum when we replace the abelian group A with a Mackey functor.

Theorem 2.34. *If $\underline{M}$ is a Mackey functor, then there exists an Eilenberg-MacLane G-spectrum $H\underline{M}$ with the property that*

$$\underline{\pi}_n(H\underline{M}) = \begin{cases} \underline{M} & \textit{if } n = 0, \\ 0 & \textit{otherwise,} \end{cases}$$

and $H\underline{M}$ is unique up to isomorphism in the homotopy category $ho\mathcal{S}p^G$ of G-spectra.

Proof. See [6, Theorem 5.3]. □

Note here that the homotopy category $ho\mathcal{S}p^G$ is obtained from the category $\mathcal{S}p^G$ of G-spectra by inverting all weak equivalences, where a map $X \to Y$ in $\mathcal{S}p^G$ is defined to be a weak equivalence if it induces an isomorphism $\pi_k^H(X) \to \pi_k^H(Y)$ on all the integer graded homotopy groups for each subgroup H of G.

If $\underline{M}$ is a Green functor, then $H\underline{M}$ is a homotopy commutative ring spectrum. Note that Theorem 2.34 only gives us the $\mathbb{Z}$-graded homotopy Mackey functors of $H\underline{M}$, not the $RO(G)$-graded homotopy Mackey functors. Indeed, the main problem of this thesis is in determining the $RO(G)$-graded homotopy Mackey functors of the Eilenberg-MacLane G-spectrum $H\underline{\mathbb{F}_2}$ where $G = C_2 \times C_2$ and $\underline{\mathbb{F}_2}$ is the constant Mackey functor associated to the field $\mathbb{F}_2$ with two elements as in Example 2.16.

Furthermore, we have that the Eilenberg-MacLane G-spectrum $H\underline{M}$ represents *Bredon homology* and *Bredon cohomology* with coefficients in the Mackey functor $\underline{M}$. Just as singular homology and cohomology with coefficients in an abelian group is easier to define for CW-complexes, we will define Bredon homology and cohomology with coefficients in a Mackey functor for *G-CW complexes.*

Definition 2.35. A *G-CW complex* is a CW-complex X with an action of G that permutes cells of the same dimension and has equivariant attaching maps. In particular, the set of n-cells of X forms a G-set which is a disjoint union of orbits, i.e. the set of n-cells is a disjoint union of *equivariant cells* (or *G-cells*) of the form $G/H_+ \wedge D^n$ for H a subgroup of G where G acts trivially on the disk D^n. A *G-CW spectrum* is the suspension spectrum of a G-CW complex.

The following is an important class of examples, and explicit G-CW structures for these will be given in later chapters for the groups $G = C_2$ and $G = C_2 \times C_2$.

Example 2.36. If V is an actual G-representation, then the representation sphere S^V is a G-CW complex.

An important fact regarding G-CW complexes is that the Whitehead theorem from non-equivariant homotopy theory generalises to the equivariant setting.

Theorem 2.37. *If X and Y are G-CW complexes, then a G-map $f\colon X \to Y$ is an equivariant homotopy equivalence if and only if the induced maps $f^H\colon X^H \to Y^H$ on the fixed-point spaces for all subgroups H of G are ordinary homotopy equivalences.*

Note that an equivariant homotopy equivalence as in the statement of Theorem 2.37 is defined as a homotopy equivalence where all homotopies are G-equivariant. The condition that a G-map $f\colon X \to Y$ induces an isomorphism on all the fixed-point spaces is precisely the condition for a morphism in $\mathcal{T}^G$ to be a weak equivalence with respect to the *Bredon model structure* on $\mathcal{T}^G$. This is equivalent to the condition that f induces an isomorphism $\pi_*^H(X) \to \pi_*^H(Y)$ on all the equivariant homotopy groups by the ordinary Whitehead theorem and that by definition

$$\pi_*^H(X) := \pi_*(X^H) \text{ and } \pi_*^H(Y) := \pi_*(Y^H).$$

Hence, we can re-state Theorem 2.37 as a map of G-CW complexes is an equivariant equivalence if and only if it is a Bredon weak equivalence. We now return to our goal of understanding the Bredon homology and cohomology of a G-CW complex with coefficients in a Mackey functor, and for simplicity we will assume that our G-CW complexes are of *finite type*, i.e. the set of n-cells for each n is a finite G-set.

Definition 2.38. Let X be a G-CW complex and $\underline{M}$ an arbitrary Mackey functor. Then, the *Bredon chain complex* $C_*(X;\underline{M})$ is a chain complex of Mackey functors where the Mackey functor $C_n(X;\underline{M})$ is defined by

$$C_n(X;\underline{M})(G/H) = \underline{M}\left(G/H \times \coprod_{\alpha_n} G/H_{\alpha_n}\right),$$

where $\coprod_{\alpha_n} G/H_{\alpha_n}$ is the G-set of n-cells in the G-CW complex X with the evident transfer and restriction maps induced by the transfer and restriction maps in the Mackey functor $\underline{M}$. The Mackey functors forming the *Bredon cochain complex* $C^*(X;\underline{M})$ are equal to the Mackey functors forming the Bredon chain complex as above. The boundary and coboundary maps in the Bredon chain and cochain complexes are induced by the cofiber sequence

$$X^{n-1}/X^{n-2} \to X^n/X^{n-2} \to X^n/X^{n-1}$$

as in the ordinary cellular chain and cochain complexes for the underlying CW-complex X, where X^k denotes the k-skeleton of X. The *Bredon homology* $\underline{H}_*(X;\underline{M})$ and *Bredon cohomology* $\underline{H}^*(X;\underline{M})$ of X with coefficients in the Mackey functor $\underline{M}$ is the homology and cohomology of the Bredon chain and cochain complexes respectively.

We will be primarily interested in taking coefficients in the constant Mackey functor $\underline{\mathbb{F}_2}$, and in this case (and indeed for the constant Mackey functor associated to any abelian group) the Bredon homology and cohomology Mackey functors can be alternatively calculated as follows. If X is a G-CW complex (or more generally a G-CW spectrum), then the ordinary cellular chain complex of the underlying CW-complex (with coefficients in $\mathbb{F}_2$) is in fact a chain complex of $\mathbb{F}_2[G]$-modules since the set of n-cells is a G-set. Each of these $\mathbb{F}_2[G]$-modules has a corresponding fixed-point Mackey functor as in Example 2.17, and we therefore get a chain complex of Mackey functors whose boundary maps on each level of our G-Mackey functors is induced by the cellular boundary map on the bottom level. The Bredon homology $\underline{H}_*(X;\underline{\mathbb{F}_2})$ is then the homology of this chain complex of fixed-point Mackey functors. The Bredon cohomology $\underline{H}^*(X;\underline{\mathbb{F}_2})$ is the homology of the chain complex of fixed-point Mackey functors corresponding to the $\mathbb{F}_2$-dual of the above cellular chain complex of $\mathbb{F}_2[G]$-modules. We will see examples of this calculation for the groups $G = C_2$ and $G = C_2 \times C_2$ in Chapters 3 and 4 respectively.

Furthermore, notice that the Bredon homology and cohomology Mackey functors of a G-CW complex X with coefficients in $\underline{\mathbb{F}_2}$ (or any constant Mackey functor) are cohomological Mackey functors as they are given by the homology of a chain complex of cohomological Mackey functors, recalling by Proposition 2.23 that fixed-point Mackey functors are cohomological. Since Bredon homology with coefficients in an arbitrary Mackey functor $\underline{M}$ is represented by the equivariant Eilenberg-MacLane spectrum $H\underline{M}$, we have that

$$\underline{H}_*(X;\underline{M}) \cong \underline{\pi}_*(X \wedge H\underline{M}).$$

Note that we can pass between $\mathbb{Z}$-graded and $RO(G)$-graded homotopy Mackey functors by smashing with (virtual) representation spheres and using the suspension isomorphism, for example if $n \in \mathbb{Z}$ and $V \in RO(G)$ then

$$\underline{\pi}_{n-V}(X \wedge H\underline{M}) \cong \underline{\pi}_n(S^V \wedge X \wedge H\underline{M}).$$

In particular, as will be our focus in later chapters, in order to compute the $RO(G)$-graded homotopy $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ it suffices to compute the $\mathbb{Z}$-graded Bredon homology $\underline{H}_*(S^V;\underline{\mathbb{F}_2}) \cong \underline{\pi}_*(S^V \wedge H\underline{\mathbb{F}_2})$ of virtual representation spheres via chain complexes of fixed-point Mackey functors.

Chapter 3

C_2-equivariant stable homotopy theory

In this chapter we will focus on the group $G = C_2$, and give a number of computations that will be generalised to the non-cyclic group $C_2 \times C_2$ in Chapter 4. In particular, we will compute the structure of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ as an $RO(C_2)$-graded Green functor, where $H\underline{\mathbb{F}_2}$ is the C_2-equivariant Eilenberg-MacLane spectrum corresponding to the constant Mackey functor $\underline{\mathbb{F}_2}$. We also introduce the concept of Anderson duality and explain why the additive structure of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ comes in two symmetric pieces as seen in Figure 3.1.

3.1 The $RO(C_2)$-graded homotopy Mackey functors

Throughout this chapter we let $G = C_2$ be the cyclic group of order two unless stated otherwise. We will write $C_2 = \{1, t\}$ so that t denotes the non-trivial element of C_2 with $t^2 = 1$. Note that C_2 has exactly two distinct one-dimensional real irreducible representations, which we will denote by 1 and σ. Here 1 is the trivial one-dimensional representation (where C_2 acts trivially on $\mathbb{R}$) and σ denotes the sign representation whereby the non-trivial element t of C_2 acts on $\mathbb{R}$ by sending a real number to its negative. Hence, we have that the orthogonal representation ring of Definition 2.30 is given by $RO(C_2) = \mathbb{Z}\{1, \sigma\}$, which is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ as an abelian group. The regular representation of C_2 is given by

$$\rho = 1 + \sigma.$$

The group $G = C_2$ has only two subgroups, so if $\underline{M}$ is an arbitrary C_2-Mackey functor then we can depict $\underline{M}$ via the diagram

$$\begin{array}{c} \underline{M}(C_2/C_2) \\ \downarrow \uparrow \\ \underline{M}(C_2/e), \end{array}$$

which is called a *Lewis diagram*, first introduced in [27]. In this section the Mackey functors of interest will have only either an $\mathbb{F}_2$ or 0 at the top and bottom levels, so the trivial Weyl group actions will not be drawn on our C_2-Mackey functors.

Example 3.1. The constant Mackey functor associated to $\mathbb{F}_2$ as in Example 2.16 has Lewis diagram

$$\underline{\mathbb{F}_2} = \begin{array}{c} \mathbb{F}_2 \\ {\scriptstyle 1}\downarrow \uparrow{\scriptstyle 0} \\ \mathbb{F}_2, \end{array}$$

and the *dual* of the constant Mackey functor has Lewis diagram

$$\underline{\mathbb{F}_2}^* = \begin{array}{c} \mathbb{F}_2 \\ {\scriptstyle 0}\downarrow \uparrow{\scriptstyle 1} \\ \mathbb{F}_2. \end{array}$$

Notice that the transfer and restriction maps of the dual constant Mackey functor are precisely the restriction and transfer maps of the constant Mackey functor respectively. Rather than simply drawing the Lewis diagram for $\underline{\mathbb{F}_2}^*$, we make the following more general definition from which the definition of the dual constant Mackey functor $\underline{\mathbb{F}_2}^*$ follows.

Definition 3.2. Let $\underline{M}$ be a C_2-Mackey functor with transfer and restriction maps $\mathrm{Tr}_e^{C_2}$ and $\mathrm{Res}_e^{C_2}$ respectively. Then, we define its $\mathbb{F}_2$-*dual* (or simply *dual*) to be the C_2-Mackey functor $\underline{M}^*$ defined by $\underline{M}^*(C_2/e) = \mathrm{Hom}(\underline{M}(C_2/e), \mathbb{F}_2)$ and $\underline{M}^*(C_2/C_2) = \mathrm{Hom}(\underline{M}(C_2/C_2), \mathbb{F}_2)$ with transfer and restriction maps $(\mathrm{Tr}_e^{C_2})^*$ and $(\mathrm{Res}_e^{C_2})^*$ being the maps induced by $\mathrm{Res}_e^{C_2}$ and $\mathrm{Tr}_e^{C_2}$ respectively.

Definition 3.2 extends easily to G-Mackey functors for any finite group G. The following two Mackey functors will also show up in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$, and we will name these Mackey functors following [3, Section 3.1].

Example 3.3. The *geometric* and *free* Mackey functors are given respectively by

$$\underline{g} = \begin{array}{c} \mathbb{F}_2 \\ \downarrow\uparrow \\ 0 \end{array} \qquad \text{and} \qquad \underline{f} = \begin{array}{c} 0 \\ \downarrow\uparrow \\ \mathbb{F}_2. \end{array}$$

Notice that the geometric Mackey functor $\underline{g}$ is precisely the image of the e-Mackey functor (or C_2/C_2-Mackey functor) $\mathbb{F}_2$ under the pullback of Mackey functors $\phi^*_{C_2} : \mathrm{Mack}(C_2/C_2) \to \mathrm{Mack}(C_2)$ induced by the quotient map $\phi_{C_2} : C_2 \to C_2/C_2$ as in Definition 2.20. Furthermore, notice that $\underline{g}$ and $\underline{f}$ are self-dual in the sense that $\underline{g}^* = \underline{g}$ and $\underline{f}^* = \underline{f}$.

We are now ready to compute the additive structure $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$, and we first look at $\underline{\pi}_k \left(S^{n\sigma} \wedge H\underline{\mathbb{F}_2}\right) = \underline{\pi}_k \left(\Sigma^{n\sigma} H\underline{\mathbb{F}_2}\right)$ where $n \geq 0$.

Proposition 3.4. *The non-zero homotopy Mackey functors of $\Sigma^{n\sigma} H\underline{\mathbb{F}_2}$ for $n \geq 0$ are given by*

$$\underline{\pi}_k \left(\Sigma^{n\sigma} H\underline{\mathbb{F}_2}\right) = \begin{cases} \underline{\mathbb{F}_2} & \text{if } k = n, \\ \underline{g} & \text{if } k \in [0, n-1]. \end{cases}$$

Proof. Fix $n \geq 0$. As discussed in Chapter 2, we know that $\underline{\pi}_k(S^{n\sigma} \wedge H\underline{\mathbb{F}_2}) = \underline{H}_k(S^{n\sigma}; \underline{\mathbb{F}_2})$, and the Bredon homology may be computed via a chain complex of fixed-point Mackey functors. To do this, we first put an explicit C_2-CW structure on the (actual) representation sphere $S^{n\sigma}$. We see that $S^{n\sigma}$ has two equivariant 0-cells indexed by C_2/C_2, and a single equivariant k-cell indexed by C_2/e for each $1 \leq k \leq n$. The (reduced) cellular chain complex computing $\underline{H}_*(S^{n\sigma}; \underline{\mathbb{F}_2})(C_2/e)$, which is isomorphic to the singular homology with $\mathbb{F}_2$-coefficients $H_*(S^n; \mathbb{F}_2)$ of the underlying sphere S^n, is given by

$$\mathbb{F}_2[C_2/C_2] \xleftarrow{\nabla} \mathbb{F}_2[C_2/e] \xleftarrow{1+t} \mathbb{F}_2[C_2/e] \xleftarrow{1+t} \cdots \xleftarrow{1+t} \mathbb{F}_2[C_2/e]$$

concentrated in degrees inside the interval $[0, n]$. Taking C_2-fixed points, we then get the chain complex of Mackey functors

$$\begin{array}{ccccccccc}
\mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \cdots & \xleftarrow{0} & \mathbb{F}_2 \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} \\
\mathbb{F}_2[C_2/C_2] & \xleftarrow{\nabla} & \mathbb{F}_2[C_2/e] & \xleftarrow{1+t} & \mathbb{F}_2[C_2/e] & \xleftarrow{1+t} & \cdots & \xleftarrow{1+t} & \mathbb{F}_2[C_2/e].
\end{array}$$

Taking homology, we get the homotopy Mackey functors

$$\begin{array}{ccccc} \mathbb{F}_2 & \mathbb{F}_2 & \mathbb{F}_2 & \cdots & \mathbb{F}_2 \\ \downarrow\uparrow & \downarrow\uparrow & \downarrow\uparrow & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} \\ 0 & 0 & 0 & \cdots & \mathbb{F}_2 \end{array}$$

in degrees 0 through n. □

It is useful (for example when discussing the slice spectral sequence) to rewrite the result of Proposition 3.4 in terms of the homotopy Mackey functors of suspensions of $H\underline{\mathbb{F}_2}$ by multiples of the regular representation, which we do by taking the trivial suspension n times.

Corollary 3.5. *The non-zero homotopy Mackey functors of $\Sigma^{n\rho}H\underline{\mathbb{F}_2}$ for $n \geq 0$ are given by*

$$\underline{\pi}_k\left(\Sigma^{n\rho}H\underline{\mathbb{F}_2}\right) = \begin{cases} \underline{\mathbb{F}_2} & \text{if } k = 2n, \\ \underline{g} & \text{if } k \in [n, 2n-1]. \end{cases}$$

We now look at suspensions of $H\underline{\mathbb{F}_2}$ by negative multiples of σ (or negative multiples of the regular representation ρ). We will see later how we can alternatively compute $\underline{\pi}_k\left(\Sigma^{-n\sigma}H\underline{\mathbb{F}_2}\right)$ where $n \geq 1$ using Anderson duality.

Proposition 3.6. *The non-zero homotopy Mackey functors of $\Sigma^{-n\sigma}H\underline{\mathbb{F}_2}$ for $n \geq 2$ are given by*

$$\underline{\pi}_k\left(\Sigma^{-n\sigma}H\underline{\mathbb{F}_2}\right) = \begin{cases} \underline{\mathbb{F}_2^*} & \text{if } k = -n, \\ \underline{g} & \text{if } k \in [-n+1, -2]. \end{cases}$$

When $n = 1$ the only non-zero homotopy Mackey functor is $\underline{f}$ in degree -1.

Proof. Since $S^{-n\sigma}$ is the Spanier-Whitehead dual (see [1, Section 2.2.1]) of $S^{n\sigma}$, we have as discussed in Chapter 2 that the chain complex computing $\underline{H}_*(S^{-n\sigma};\underline{\mathbb{F}_2})(C_2/e)$ is the $\mathbb{F}_2$-dual of the chain complex from the proof of Proposition 3.4, namely the chain complex

$$\mathbb{F}_2[C_2/C_2] \xrightarrow{\Delta} \mathbb{F}_2[C_2/e] \xrightarrow{1+t} \mathbb{F}_2[C_2/e] \xrightarrow{1+t} \cdots \xrightarrow{1+t} \mathbb{F}_2[C_2/e]$$

concentrated in degrees inside the interval $[-n, 0]$. Taking C_2-fixed points, we then get the chain complex of Mackey functors

$$\begin{array}{ccccccccc} \mathbb{F}_2 & \xrightarrow{1} & \mathbb{F}_2 & \xrightarrow{0} & \mathbb{F}_2 & \xrightarrow{0} & \cdots & \xrightarrow{0} & \mathbb{F}_2 \\ {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} \\ \mathbb{F}_2[C_2/C_2] & \xrightarrow{\Delta} & \mathbb{F}_2[C_2/e] & \xrightarrow{1+t} & \mathbb{F}_2[C_2/e] & \xrightarrow{1+t} & \cdots & \xrightarrow{1+t} & \mathbb{F}_2[C_2/e]. \end{array}$$

The homology of this chain complex is given (in the case that $n \geq 2$) by

$$\begin{array}{ccccc} 0 & 0 & \mathbb{F}_2 & \cdots & \mathbb{F}_2 \\ \updownarrows & \updownarrows & \updownarrows & & {\scriptstyle 0}\updownarrows{\scriptstyle 1} \\ 0 & 0 & 0 & \cdots & \mathbb{F}_2. \end{array}$$

However, when $n = 1$ we are just left with

$$\begin{array}{cc} 0 & 0 \\ \updownarrows & \updownarrows \\ 0 & \mathbb{F}_2. \end{array}$$

□

Analogous to Corollary 3.5, we may desuspend n times in order to re-write the result of Proposition 3.6 in terms of suspensions of $H\underline{\mathbb{F}_2}$ via negative multiples of the regular representation.

Corollary 3.7. *The non-zero homotopy Mackey functors of $\Sigma^{-n\rho} H\underline{\mathbb{F}_2}$ for $n \geq 2$ are given by*

$$\underline{\pi}_k \left(\Sigma^{-n\rho} H\underline{\mathbb{F}_2}\right) = \begin{cases} \underline{\mathbb{F}_2^*} & \text{if } k = -2n, \\ \underline{g} & \text{if } k \in [-2n+1, -n-2]. \end{cases}$$

When $n = 1$ the only non-zero homotopy Mackey functor is $\underline{f}$ in degree $-n-1$.

3.2 Anderson duality

We introduce equivariant Anderson duality in the context of giving an alternative argument to deduce the homotopy Mackey functors given in Proposition 3.6. We observe the following twisting.

Proposition 3.8. *As C_2-spectra, we have that $\Sigma^4 H\underline{\mathbb{F}_2} \simeq \Sigma^{2\rho} H\underline{\mathbb{F}_2^*}$.*

Proof. It suffices to show that $\Sigma^{4-2\rho} H\underline{\mathbb{F}_2} \simeq H\underline{\mathbb{F}_2^*}$, and to do this we will show that both spectra have the same homotopy Mackey functors. We see that

$$\begin{aligned} \underline{\pi}_k(\Sigma^{4-2\rho} H\underline{\mathbb{F}_2}) &= \underline{\pi}_k(\Sigma^{2-2\sigma} H\underline{\mathbb{F}_2}) \\ &= \underline{\pi}_{k-2}(\Sigma^{-2\sigma} H\underline{\mathbb{F}_2}). \end{aligned}$$

However, by Proposition 3.6 we know that

$$\underline{\pi}_{k-2}\left(\Sigma^{-2\sigma}H\underline{\mathbb{F}_2}\right) = \begin{cases} \underline{\mathbb{F}_2^*} & \text{if } k-2=-2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we have that

$$\underline{\pi}_k(\Sigma^{4-2\rho}H\underline{\mathbb{F}_2}) = \begin{cases} \underline{\mathbb{F}_2^*} & \text{if } k=0, \\ 0 & \text{otherwise,} \end{cases}$$

which is precisely $\underline{\pi}_k(H\underline{\mathbb{F}_2^*})$. Hence, we must have that $\Sigma^{4-2\rho}H\underline{\mathbb{F}_2} \simeq H\underline{\mathbb{F}_2^*}$ by the uniqueness of Eilenberg-MacLane spectra as in Theorem 2.34. □

In the following discussion, we can take G to be an arbitrary finite group. Consider the category $H\underline{\mathbb{F}_2}$-*mod* of $H\underline{\mathbb{F}_2}$-modules, where here $\underline{\mathbb{F}_2}$ is the constant G-Mackey functor associated to $\mathbb{F}_2$. Then, we have a contravariant functor

$$I_{\mathbb{F}_2}\colon (H\underline{\mathbb{F}_2}\text{-}mod)^{op} \to H\underline{\mathbb{F}_2}\text{-}mod$$

called *Anderson duality*, where if X is an $H\underline{\mathbb{F}_2}$-module then the homotopy Mackey functors of its *Anderson dual* $I_{\mathbb{F}_2}X$ are given by

$$\underline{\pi}_V I_{\mathbb{F}_2}X = (\underline{\pi}_{-V}X)^*$$

for $V \in RO(G)$. Note that on the right-hand side we are taking the $\mathbb{F}_2$-dual of the Mackey functor $\underline{\pi}_{-V}X$ as in Definition 3.2.

Proposition 3.9. *Let $\underline{M}$ be an $\underline{\mathbb{F}_2}$-module. Then, we have that $I_{\mathbb{F}_2}H\underline{M} = H\underline{M}^*$. That is, for every $V \in RO(G)$ we have that*

$$\underline{\pi}_V(H\underline{M}^*) \cong (\underline{\pi}_{-V}H\underline{M})^*.$$

Proof. See [3, Proposition 2.9]. □

More detailed discussion on Anderson duality can be found in [7] and [16]. Note that in particular Proposition 3.9 implies that $I_{\mathbb{F}_2}H\underline{\mathbb{F}_2} = H\underline{\mathbb{F}_2^*}$. Using this, we give an alternative proof of Proposition 3.6 (in the case that $n \geq 2$).

Proof of Proposition 3.6. Let $n \geq 2$ and $k \in \mathbb{Z}$ be arbitrary. We want to compute $\underline{\pi}_k(\Sigma^{-n\sigma}H\underline{\mathbb{F}_2}) = \underline{\pi}_{k+n\sigma}(H\underline{\mathbb{F}_2})$. However, by Proposition 3.9 we have that

$$\begin{aligned}(\underline{\pi}_{k+n\sigma}H\underline{\mathbb{F}_2})^* &\cong \underline{\pi}_{-k-n\sigma}(H\underline{\mathbb{F}_2^*}) \\ &= \underline{\pi}_{-k}(\Sigma^{n\sigma}H\underline{\mathbb{F}_2^*})\end{aligned}$$

But by Proposition 3.8, we know that $\Sigma^{2-2\sigma} H\underline{\mathbb{F}_2} \simeq H\underline{\mathbb{F}_2^*}$. That is,

$$\begin{aligned} \underline{\pi}_{-k}(\Sigma^{n\sigma} H\underline{\mathbb{F}_2^*}) &\cong \underline{\pi}_{-k}(\Sigma^{2+(n-2)\sigma} H\underline{\mathbb{F}_2}) \\ &= \underline{\pi}_{-k-2}(\Sigma^{(n-2)\sigma} H\underline{\mathbb{F}_2}). \end{aligned}$$

Hence, we have shown that

$$\underline{\pi}_k(\Sigma^{-n\sigma} H\underline{\mathbb{F}_2}) \cong (\underline{\pi}_{-k-2}(\Sigma^{(n-2)\sigma} H\underline{\mathbb{F}_2}))^*,$$

and the result now follows after noticing that $\underline{g}^* = \underline{g}$. □

The Mackey functor homotopy of $H\underline{\mathbb{F}_2}$ that we have now computed is depicted in Figure 3.1 below. In particular, our above discussion of Anderson duality

Figure 3.1: $\underline{\pi}_k(\Sigma^{n\sigma} H\underline{\mathbb{F}_2})$

n \ k	-5	-4	-3	-2	-1	0	1	2	3
3						$\underline{g}$	$\underline{g}$	$\underline{g}$	$\underline{\mathbb{F}_2}$
2						$\underline{g}$	$\underline{g}$	$\underline{\mathbb{F}_2}$	
1						$\underline{g}$	$\underline{\mathbb{F}_2}$		
0						$\underline{\mathbb{F}_2}$			
-1					$\underline{f}$				
-2				$\underline{\mathbb{F}_2^*}$					
-3			$\underline{\mathbb{F}_2^*}$	$\underline{g}$					
-4		$\underline{\mathbb{F}_2^*}$	$\underline{g}$	$\underline{g}$					
-5	$\underline{\mathbb{F}_2^*}$	$\underline{g}$	$\underline{g}$	$\underline{g}$					

explains the evident symmetry between the first and third quadrants (if we ignore the free Mackey functor $\underline{f}$ which in any case is zero at the C_2/C_2 level).

3.3 The ring structure of the $RO(C_2)$-graded homotopy

Since $H\underline{\mathbb{F}_2}$ is a commutative ring spectrum, we know by Theorem 2.33 that $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is an $RO(C_2)$-graded Green functor. That is, we can write $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ as a single Mackey functor where both $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}(C_2/C_2)$ and $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}(C_2/e)$ are commutative $RO(C_2)$-graded rings, the restriction map is a ring map and the transfer map satisfies Frobenius reciprocity. Our goal in this section is to compute these ring structures explicitly.

We begin by computing the ring structure of the top level $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}(C_2/C_2)$, which as mentioned in Chapter 2 we will re-write as $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$. If we focus on the top levels of the Mackey functors in Figure 3.1, then we say that the non-zero elements in the first quadrant form the *positive cone*, and non-zero elements in the third quadrant form the *negative cone.* Now, let

$$x \in \pi_0^{C_2}(S^\sigma \wedge H\underline{\mathbb{F}_2}) \text{ and } y \in \pi_1^{C_2}(S^\sigma \wedge H\underline{\mathbb{F}_2})$$

be the generators of the two copies of $\mathbb{F}_2$ in $\pi_*^{C_2}(S^\sigma \wedge H\underline{\mathbb{F}_2})$ at degrees 0 and 1 respectively. Then, we claim that the positive cone is polynomial in x and y. Note that for example in [8] these two elements are named ρ and τ respectively.

Theorem 3.10. *The positive cone in $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$ is given by the graded polynomial ring $\mathbb{F}_2[x, y]$, where the elements x and y are defined as above.*

Proof. Given that we already know the additive structure of $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$ from Section 3.1, it suffices to show that if z generates a copy of $\mathbb{F}_2$ in the positive cone, then both zx and zy are non-zero i.e. that they also generate copies of $\mathbb{F}_2$ in the positive cone. So, let $n \geq 1$ be arbitrary and suppose that z is the generator of $\pi_k^{C_2}(S^{n\sigma} \wedge H\underline{\mathbb{F}_2})$ for some $0 \leq k \leq n$. The following diagram shows the top row of the chain complex of Mackey functors (as constructed in the proof of Proposition 3.4) computing $\pi_k^{C_2}(S^{n\sigma} \wedge H\underline{\mathbb{F}_2})$ as well as the class z:

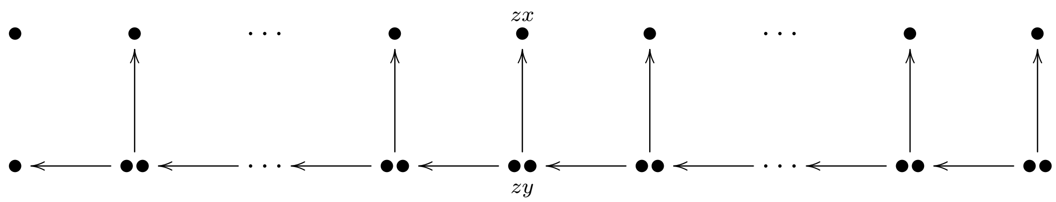
$$\bullet \quad \bullet \quad \cdots \quad \bullet \quad \overset{z}{\bullet} \quad \bullet \quad \cdots \quad \bullet \quad \bullet$$

Here each $\bullet$ represents a copy of $\mathbb{F}_2$, and no differentials are drawn as they are all zero. Now, the ring multiplication in the positive cone is induced by

$$\pi_i^{C_2}(S^{n\sigma} \wedge H\underline{\mathbb{F}_2}) \otimes \pi_j^{C_2}(S^{n'\sigma} \wedge H\underline{\mathbb{F}_2}) \to \pi_{i+j}^{C_2}(S^{(n+n')\sigma} \wedge H\underline{\mathbb{F}_2})$$

whereby we take the tensor product of the $\mathbb{F}_2[C_2]$-modules forming the C_2/e levels of the chain complexes of Mackey functors computing $\underline{\pi}_i(S^{n\sigma} \wedge H\underline{\mathbb{F}_2})$ and

$\underline{\pi}_j(S^{n'\sigma} \wedge H\underline{\mathbb{F}_2})$ and then take C_2-fixed points. We want to multiply z by x and by y, so we will be tensoring with the chain complex computing $\underline{\pi}_*(S^\sigma \wedge H\underline{\mathbb{F}_2})$ (i.e. we take $n' = 1$). Focusing on the top level of our Mackey functors (i.e. taking C_2-fixed points of the tensor product), we obtain the following double complex:



Here each $\bullet\bullet$ represents a copy of $\mathbb{F}_2^2$, which is the C_2-fixed points of the $\mathbb{F}_2$-module $\mathbb{F}_2[C_2/e \times C_2/e]$. Note that we have an explicit decomposition of the product $C_2/e \times C_2/e$ into two free transitive C_2-orbits $C_2/e \sqcup C_2/e$ as follows. For ease of notation, we write elements in $C_2/e \times C_2/e$ as $z_{ij} = (z_i, z_j)$ for $0 \le i, j \le 1$ where $z_0 = 1$ and $z_1 = t$. Then, the two free transitive C_2-orbits are $\{z_{00}, z_{11}\}$ and $\{z_{10}, z_{01}\}$. Also, note that each single $\bullet$ in the above diagram is the C_2-fixed points of the $\mathbb{F}_2[C_2]$-module $\mathbb{F}_2[C_2/e \times C_2/C_2] \cong \mathbb{F}_2[C_2/e]$, except at degree 0 which is the C_2-fixed points of $\mathbb{F}_2[C_2/C_2]$.

Furthermore, each differential $\bullet\bullet \to \bullet$ in the above double complex is the co-diagonal map ∇, and each differential $\bullet\bullet \to \bullet\bullet$ is given by the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. To see this, since we are taking the tensor product we know that each differential

$$\mathbb{F}_2[C_2/e \times C_2/e]^{C_2} \to \mathbb{F}_2[C_2/e \times C_2/C_2]^{C_2}$$

is given by applying the map $\mathrm{id} \times (z_0 \mapsto 1, z_1 \mapsto 1)$ and taking C_2-fixed points. So, applying this map to each of our four basis elements $\{z_{00}, z_{11}\} \cup \{z_{10}, z_{01}\}$ of $\mathbb{F}_2[C_2/e \times C_2/e]$, we see that

$$\begin{aligned} z_{00} &\mapsto z_0, & z_{10} &\mapsto z_1, \\ z_{11} &\mapsto z_1, & z_{01} &\mapsto z_0. \end{aligned}$$

Thus, after taking C_2-fixed points we see that the two basis elements $z_{00} + z_{11}$ and $z_{10} + z_{01}$ of $\mathbb{F}_2[C_2/e \times C_2/e]^{C_2}$ map to $z_0 + z_1$, so indeed we obtain the co-diagonal map ∇. Similarly, each differential

$$\mathbb{F}_2[C_2/e \times C_2/e]^{C_2} \to \mathbb{F}_2[C_2/e \times C_2/e]^{C_2}$$

is given by applying the map $(1 + t) \times \mathrm{id}$ and taking C_2-fixed points. Applying this map to each of our four basis elements of $\mathbb{F}_2[C_2/e \times C_2/e]$, we see that

$$\begin{aligned} z_{00} &\mapsto z_{00} + z_{10}, & z_{10} &\mapsto z_{10} + z_{00}, \\ z_{11} &\mapsto z_{11} + z_{01}, & z_{01} &\mapsto z_{01} + z_{11}. \end{aligned}$$

Therefore, after taking C_2-fixed points we see that each of the two basis elements $z_{00} + z_{11}$ and $z_{10} + z_{01}$ of $\mathbb{F}_2[C_2/e \times C_2/e]^{C_2}$ map to the sum of the two basis elements of $\mathbb{F}_2[C_2/e \times C_2/e]^{C_2}$, so indeed we are left with the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

It remains to show that the elements in the double complex representing zx and zy are non-zero in homology. To do this, first suppose that $k \geq 1$. Then, as indicated on the diagram of our double complex above, the product zx is represented by the sum $z_0 + z_1 \in \mathbb{F}_2[C_2/e \times C_2/C_2]^{C_2} \cong \mathbb{F}_2[C_2/e]^{C_2}$ as z is represented by $z_0 + z_1 \in \mathbb{F}_2[C_2/e]^{C_2}$ in our original chain complex computing $\pi_k^{C_2}(S^{n\sigma} \wedge H\underline{\mathbb{F}_2})$, and $x \in \pi_0^{C_2}(S^\sigma \wedge H\underline{\mathbb{F}_2})$ is represented by $1 \in \mathbb{F}_2[C_2/C_2]^{C_2}$. However, we have that $y \in \pi_1^{C_2}(S^\sigma \wedge H\underline{\mathbb{F}_2})$ is represented by $z_0 + z_1 \in \mathbb{F}_2[C_2/e]^{C_2}$, so the product zy is represented by $z_{00} + z_{11} + z_{10} + z_{01} \in \mathbb{F}_2[C_2/e \times C_2/e]^{C_2}$, i.e. by the sum of the two basis elements of $\mathbb{F}_2[C_2/e \times C_2/e]^{C_2}$.

Therefore, we have that both zx and zy are in the kernel of the total differential of the double complex (which is defined to be the sum of the horizontal and vertical differentials), and thus they are homology classes. However, even though both zx and zy are in the image of either a horizontal or vertical differential (unless $k = n$ in which case zy is not in the image of a horizontal or vertical differential), they are not in the image of the total differential so they are non-zero homology classes.

Note that if $k = 0$, then zy is represented by $z_0 + z_1 \in \mathbb{F}_2[C_2/e \times C_2/C_2]^{C_2} \cong \mathbb{F}_2[C_2/e]^{C_2}$ which is still in the kernel and not in the image of the total differential, and zx is represented by $1 \in \mathbb{F}_2[C_2/C_2 \times C_2/C_2]^{C_2}$ and is in the kernel of the total differential and in particular not in the image of the horizontal or vertical differentials as depicted on our earlier diagram of this double complex. $\square$

Now that we understand the positive cone, we turn to the negative cone in $\pi_\star^{C_2} H\underline{\mathbb{F}_2}$. Since the free Mackey functor $\underline{f}$ has $\underline{f}(C_2/C_2) = 0$ (as in Example 3.3), we see from Figure 3.1 that the smallest positive n such that $\pi_*^{C_2}(S^{-n\sigma} \wedge H\underline{\mathbb{F}_2})$ has a non-zero class is $n = 2$. The homotopy $\pi_*^{C_2}(S^{-2\sigma} \wedge H\underline{\mathbb{F}_2})$ is concentrated in degree -2 and we let

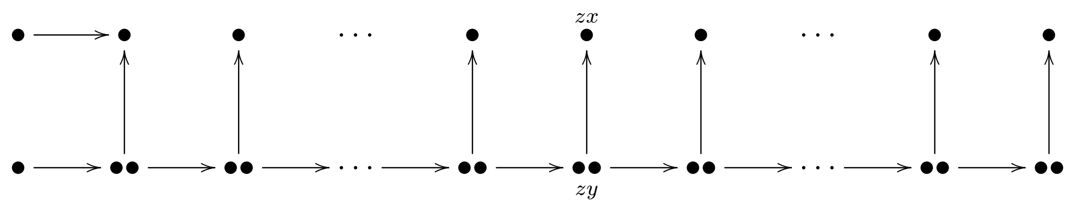
$$\theta \in \pi_{-2}^{C_2}(S^{-2\sigma} \wedge H\underline{\mathbb{F}_2})$$

be the generator of the single copy of $\mathbb{F}_2$ in this degree. We claim that this class θ is 'infinitely divisible' by x and y in the sense of the following theorem.

Theorem 3.11. *The negative cone in* $\pi_\star^{C_2} H\underline{\mathbb{F}_2}$ *is given by* $\frac{\mathbb{F}_2[x,y]}{(x^\infty, y^\infty)}\{\theta\}$ *with the class* θ *defined as above. That is, any non-zero class in the negative cone is represented by some element* $\frac{\theta}{x^i y^j} \in \frac{\mathbb{F}_2[x,y]}{(x^\infty, y^\infty)}\{\theta\}$ *in the sense that we get back the* *class θ after multiplying by the class x^iy^j in the positive cone, and we get zero when we multiply by any class in the positive cone that does not divide x^iy^j.*

Proof. Again, we know the additive structure of $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$ from Section 3.1, so it suffices to show that if z generates a copy of $\mathbb{F}_2$ in the negative cone, then zx and zy are non-zero unless they are forced to be zero by degree reasons (i.e. if multiplying by x or y pushes us into a degree whose corresponding homotopy group is zero). So, let $n \geq 3$ be arbitrary, and suppose that z is the generator of $\pi_k^{C_2}(S^{-n\sigma} \wedge H\underline{\mathbb{F}_2})$ for some $-n \leq k \leq -2$. Note that we ignore the case $n = 2$ as $\theta x = \theta y = 0$ by degree reasons. Now, the following diagram shows the top row of the chain complex of fixed-point Mackey functors computing $\pi_k^{C_2}(S^{-n\sigma} \wedge H\underline{\mathbb{F}_2})$ together with the class z:

Again each $\bullet$ represents a copy of $\mathbb{F}_2$ and an arrow is drawn in the chain complex if and only if the corresponding differential is zero. As in the proof of Theorem 3.10, we now want to tensor with the chain complex computing $\pi_*^{C_2}(S^\sigma \wedge H\underline{\mathbb{F}_2})$ which results in the following double complex:



All the horizontal and vertical differentials in this double complex were computed in the proof of Theorem 3.10 except for the differential $\bullet \to \bullet\bullet$ which we claim is the diagonal map Δ. Indeed, this horizontal differential

$$\mathbb{F}_2[C_2/C_2 \times C_2/e]^{C_2} \to \mathbb{F}_2[C_2/e \times C_2/e]^{C_2}$$

is given by applying the map $(1 \mapsto z_0 + z_1) \times \mathrm{id}$ and taking C_2-fixed points, noting that we use the same notation as in the proof of Theorem 3.10. Now, applying this map to the basis $\{z_0, z_1\}$ of $\mathbb{F}_2[C_2/e]$, we see that

$$z_0 \mapsto z_{00} + z_{10} \text{ and } z_1 \mapsto z_{01} + z_{11}.$$

Hence, after taking C_2-fixed points we get that the basis element $z_0 + z_1$ of $\mathbb{F}_2[C_2/e]^{C_2}$ maps to the diagonal element $z_{00} + z_{11} + z_{10} + z_{01}$ of $\mathbb{F}_2[C_2/e \times C_2/e]^{C_2}$, so this horizontal differential is indeed the diagonal map.

Now, as in the proof of Theorem 3.10 we have that zx is represented by $z_0 + z_1 \in \mathbb{F}_2[C_2/e \times C_2/C_2]^{C_2} \cong \mathbb{F}_2[C_2/e]^{C_2}$ and zy is represented by the diagonal element $z_{00} + z_{11} + z_{10} + z_{01} \in \mathbb{F}_2[C_2/e \times C_2/e]^{C_2}$, so both zx and zy are in the kernel of the total differential, i.e. are indeed homology classes. Suppose first that $-n < k < -2$. By the same argument as in the proof of Theorem 3.10, we have that both zx and zy are not in the image of the total differential (even though they are in the image of the vertical and horizontal differential respectively) so therefore represent non-zero classes in homology. This same argument works if $k = -n$ and we are looking at zy, or if $k = -2$ and we are looking at zx.

However, if $k = -n$ then by degree reasons we know that zx is zero in homology, which can also be seen by looking at the above double complex as it is indeed in the image of the total differential. If $k = -2$, then again by degree reasons we know that zy is zero in homology, which can also be seen in the above double complex as it is homologous to a class which is in the image of the total differential. □

Figure 3.2 below depicts the top levels of the Mackey functors in Figure 3.1 and what we know at this point about the ring structure of $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$, whilst highlighting the duality between the positive and negative cones.

However, in order to complete our understanding of the ring structure of $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$, we need to know what happens when we multiply two elements in the negative cone.

Proposition 3.12. *The product of any two elements in the negative cone of* $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$ *is zero.*

Proof. Consider two arbitrary elements $\frac{\theta}{x^i y^j}$ and $\frac{\theta}{x^k y^\ell}$ in the negative cone where $i, j, k, \ell \geq 0$. Then, we want to show that $\frac{\theta}{x^i y^j} \cdot \frac{\theta}{x^k y^\ell} = 0$. To see this, suppose for the sake of a contradiction that this product is non-zero. By our description of the negative cone as seen in Figure 3.2, we have that

$$\frac{\theta}{x^i y^j} \in \pi_{-2-j}^{C_2}(S^{(-2-(i+j))\sigma} \wedge H\underline{\mathbb{F}_2})$$

and

$$\frac{\theta}{x^k y^\ell} \in \pi_{-2-\ell}^{C_2}(S^{(-2-(k+\ell))\sigma} \wedge H\underline{\mathbb{F}_2}).$$

Therefore, since we assume that their product is non-zero, we have that

$$\frac{\theta}{x^i y^j} \cdot \frac{\theta}{x^k y^\ell} = \frac{\theta}{x^{i+k} y^{j+\ell+2}} \in \pi_{-2-(j+\ell+2)}^{C_2}(S^{(-2-(i+k+j+\ell+2))\sigma} \wedge H\underline{\mathbb{F}_2}).$$

Figure 3.2: $\pi_k^{C_2}(S^{n\sigma} \wedge H\underline{\mathbb{F}_2})$

$n \uparrow$ \ $k \rightarrow$	-5	-4	-3	-2	-1	0	1	2	3
3						x^3	x^2y	xy^2	y^3
2						x^2	xy	y^2	
1						x	y		
0						1			
-1									
-2				θ					
-3			$\frac{\theta}{y}$	$\frac{\theta}{x}$					
-4		$\frac{\theta}{y^2}$	$\frac{\theta}{xy}$	$\frac{\theta}{x^2}$					
-5	$\frac{\theta}{y^3}$	$\frac{\theta}{xy^2}$	$\frac{\theta}{x^2y}$	$\frac{\theta}{x^3}$					

Now, by Theorem 3.11 we know that

$$\theta = \frac{\theta}{x^i y^j} \cdot x^i y^j = \frac{\theta}{x^k y^\ell} \cdot x^k y^\ell,$$

and hence by our above assumption we have that

$$\begin{aligned} \theta^2 &= \frac{\theta}{x^i y^j} \cdot \frac{\theta}{x^k y^\ell} \cdot x^i y^j x^k y^\ell \\ &= \frac{\theta}{x^{i+k} y^{j+\ell+2}} x^{i+k} y^{j+\ell} \\ &= \frac{\theta}{y^2}. \end{aligned}$$

However, we have that

$$\frac{\theta}{y^2} \cdot y = \frac{\theta}{y}$$

is non-zero, but

$$\theta^2 y = \theta(\theta y) = 0$$

as $\theta y = 0$ for degree reasons, so we have indeed reached a contradiction. $\square$

Now that we have derived the complete ring structure of $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$, the $RO(C_2)$-graded Green functor structure of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is a corollary. In what follows, we refer to the positive and negative cones of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ to be the first and third (minus the free Mackey functor $\underline{f} = \underline{\pi}_{-1}(\Sigma^{-\sigma} H\underline{\mathbb{F}_2})$) quadrants respectively of Figure 3.1, i.e. these terms now refer to the whole Mackey functors, not just the top levels of the Mackey functors.

Corollary 3.13. *The positive cone in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is given by the Mackey functor of $RO(C_2)$-graded rings*

$$\begin{array}{c} \mathbb{F}_2[x,y] \\ \pi \big\downarrow \big\uparrow 0 \\ \mathbb{F}_2[x,y]/(x), \end{array}$$

where $\pi\colon \mathbb{F}_2[x,y] \to \mathbb{F}_2[x,y]/(x)$ is the quotient ring map.

Proof. Let $n \geq 0$ be arbitrary. Then, by Theorem 3.10 we know that the homology of the chain complex of fixed-point Mackey functors computing $\underline{\pi}_*(\Sigma^{n\sigma} H\underline{\mathbb{F}_2})$ as constructed in the proof of Proposition 3.4 is given by

$$\begin{array}{ccccc} \mathbb{F}_2\{x^n\} & \mathbb{F}_2\{x^{n-1}y\} & \mathbb{F}_2\{x^{n-2}y^2\} & \cdots & \mathbb{F}_2\{y^n\} \\ \big\downarrow \big\uparrow & \big\downarrow \big\uparrow & \big\downarrow \big\uparrow & & 1 \big\downarrow \big\uparrow 0 \\ 0 & 0 & 0 & \cdots & \mathbb{F}_2\{\mathrm{Res}_e^{C_2}(y^n)\}, \end{array}$$

where here the notation $\mathbb{F}_2\{a\}$ means a copy of $\mathbb{F}_2$ generated by a, and we identify $\mathrm{Res}_e^{C_2}(y^n)$ with y^n as the restriction map in degree n is the identity map. $\square$

Remark 3.14. The Mackey functor given in the statement of Corollary 3.13 is an $RO(C_2)$-graded Green functor, noting that Frobenius reciprocity is of course satisfied as the transfer map is zero. A similar remark holds for Corollary 3.15 below.

We now look at the negative cone of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$, and focus on the subring $\frac{\mathbb{F}_2[x,y]}{(x^\infty, y^\infty)}\{\theta\}$ of $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$, where we recall that this subring has trivial multiplication. Consider the map

$$\frac{\mathbb{F}_2[x,y]}{(x^\infty, y^\infty)}\{\theta\} \xrightarrow{x} \frac{\mathbb{F}_2[x,y]}{(x^\infty, y^\infty)}\{\theta\}$$

given by multiplication by x. The kernel of this map can be thought of as the $\mathbb{F}_2$-linear span of the collection of all θ/y^i for $i \geq 0$ inside $\frac{\mathbb{F}_2[x,y]}{(x^\infty, y^\infty)}\{\theta\}$, and we denote the kernel by $\ker(x)$.

Corollary 3.15. *The negative cone in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is given by the Mackey functor of $RO(C_2)$-graded rings*

$$\begin{array}{c} \frac{\mathbb{F}_2[x,y]}{(x^\infty,y^\infty)}\{\theta\} \\ {\scriptstyle 0}\big\downarrow \big\uparrow{\scriptstyle i} \\ \frac{\mathbb{F}_2[x,y]}{(x^\infty,y^\infty)}\{\theta\} \cap ker(x), \end{array}$$

where $i\colon \frac{\mathbb{F}_2[x,y]}{(x^\infty,y^\infty)}\{\theta\} \cap ker(x) \to \frac{\mathbb{F}_2[x,y]}{(x^\infty,y^\infty)}\{\theta\}$ is the inclusion map.

Proof. Let $n \geq 2$ be arbitrary. Then, by Theorem 3.11 we have that the homology of the chain complex of fixed-point Mackey functors computing $\underline{\pi}_*(\Sigma^{-n\sigma} H\underline{\mathbb{F}_2})$ as constructed in the first proof of Proposition 3.6 is given by

$$\begin{array}{cccccc} 0 & 0 & \mathbb{F}_2\{\frac{\theta}{x^{n-2}}\} & \mathbb{F}_2\{\frac{\theta}{x^{n-3}y}\} & \cdots & \mathbb{F}_2\{\frac{\theta}{y^{n-2}}\} \\ \big\downarrow\big\uparrow & \big\downarrow\big\uparrow & \big\downarrow\big\uparrow & \big\downarrow\big\uparrow & & {\scriptstyle 0}\big\downarrow\big\uparrow{\scriptstyle 1} \\ 0 & 0 & 0 & 0 & \cdots & \mathbb{F}_2\{\frac{\theta}{y^{n-2}}\}, \end{array}$$

where we are identifying the generator of the $\mathbb{F}_2$ at the bottom level of the Mackey functor in degree $-n$ with θ/y^{n-2} as the transfer map is the identity. □

Chapter 4

The Klein four homology of a point

In this chapter we discuss the structure of the Green functor $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ for the group $G = C_2 \times C_2$. In particular, we look at the ring structure of the top level $\pi_\bigstar^G H\underline{\mathbb{F}_2}$ (which can be thought of as the G-equivariant homology of a point), and we give a complete algebraic description of the whole Mackey functor structure of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$. The additive structure of the top level $\pi_\bigstar^G H\underline{\mathbb{F}_2}$ is computed in [2] in the form of Poincaré series for the dimensions of the corresponding $\mathbb{F}_2$-vector spaces, and we show how to derive these Poincaré series by constructing explicit G-CW structures on G-representation spheres and iteratively using the spectral sequence of a double complex. This method can also be used to derive $\pi_\bigstar^G H\underline{\mathbb{Z}}$, and we discuss this in Section 4.8 as well as how we can instead use the Bockstein spectral sequence to compute the homology with integer coefficients.

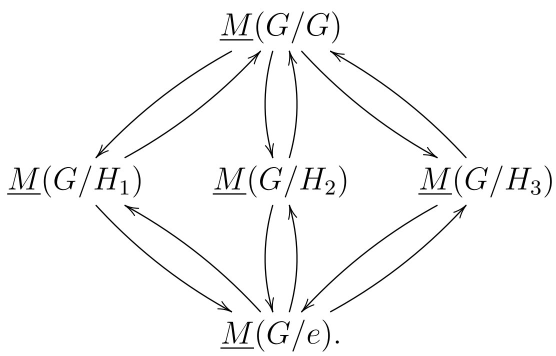
4.1 $(C_2 \times C_2)$-Mackey functors and representations

Throughout this section, and indeed throughout this chapter, we let $G = C_2 \times C_2$ with presentation

$$G = \langle t_1, t_2 \,|\, t_1^2 = t_2^2 = (t_1 t_2)^2 = 1 \rangle,$$

and we let $t_3 = t_1 t_2$. We can think of the identity element in $C_2 \times C_2$ as $(1,1)$, t_1 as $(t,1)$, t_2 as $(1,t)$ and t_3 as (t,t), with $C_2 \times C_2 = \{1,t\} \times \{1,t\}$ following the notation of Chapter 3. The Klein four group has three non-trivial C_2-subgroups generated by t_1, t_2 and t_3 respectively, and we let $H_1 = \langle t_1 \rangle$, $H_2 = \langle t_2 \rangle$ and

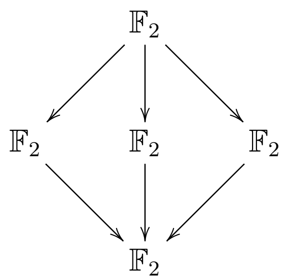
$H_3 = \langle t_3 \rangle$. Given these notational conventions, if $\underline{M}$ is a G-Mackey functor then we will depict $\underline{M}$ by the Lewis diagram



Note that the Weyl group actions are not drawn on the above diagram as for the most part these actions will be trivial for the Mackey functors we will be considering, where the values of the Mackey functors on each orbit space will be $\mathbb{F}_2$-vector spaces. In particular, the Weyl group actions are trivial for the homotopy Mackey functors in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$. Furthermore, we will not draw any transfer or restriction maps on our Mackey functors that are zero, so if an arrow $\mathbb{F}_2 \to \mathbb{F}_2$ is drawn on a Mackey functor then we know that it represents the identity map.

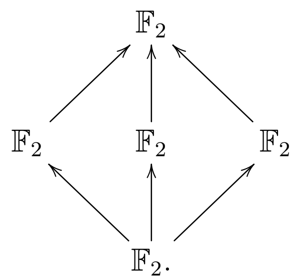
Example 4.1. The Lewis diagrams for the constant Mackey functor associated to $\mathbb{F}_2$ and its dual are given by

$\underline{\mathbb{F}_2} =$



and

$\underline{\mathbb{F}_2^*} =$



In particular, all restriction maps are the identity and all transfer maps are zero in the Mackey functor $\underline{\mathbb{F}_2}$, and the reverse is true for the Mackey functor $\underline{\mathbb{F}_2^*}$. The definition of the dual constant Mackey functor also follows Definition 3.2 for $(C_2 \times C_2)$-Mackey functors.

As an abelian group, we have that $RO(G) \cong \mathbb{Z}^4$. In particular, $C_2 \times C_2$ has precisely three distinct non-trivial one-dimensional real representations, which we will call σ_1, σ_2 and σ_3 so that the regular representation is given by

$$\rho = 1 + \sigma_1 + \sigma_2 + \sigma_3.$$

More specifically, let i_{H_1}, i_{H_2} and i_{H_3} denote the inclusions of the C_2-subgroups H_1, H_2 and H_3 into G respectively. Then, by pre-composing representations of G with these three inclusions, we obtain three induced maps $i_{H_1}^*\colon RO(G) \to RO(H_1)$, $i_{H_2}^*\colon RO(G) \to RO(H_2)$ and $i_{H_3}^*\colon RO(G) \to RO(H_3)$. Letting σ be the sign representation of C_2, the distinct irreducible representations σ_1, σ_2 and σ_3 satisfy

$$i_{H_1}^*(\sigma_1) = 1,\ i_{H_1}^*(\sigma_2) = \sigma \text{ and } i_{H_1}^*(\sigma_3) = \sigma,$$

$$i_{H_2}^*(\sigma_1) = \sigma,\ i_{H_2}^*(\sigma_2) = 1 \text{ and } i_{H_2}^*(\sigma_3) = \sigma,$$

$$i_{H_3}^*(\sigma_1) = \sigma,\ i_{H_3}^*(\sigma_2) = \sigma \text{ and } i_{H_3}^*(\sigma_3) = 1.$$

That is, the one-dimensional real representation σ_1 is defined by H_1 acting trivially on $\mathbb{R}$ with t_2 and t_3 acting non-trivially, and σ_2 and σ_3 are defined similarly where H_2 acts trivially and H_3 acts trivially respectively.

4.2 A trigraded complex of Mackey functors

In this section, we will construct for each $(p, q, r) \in \mathbb{Z}^3$ a triple complex of Mackey functors whose homology is precisely $\underline{\pi}_*(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{F}_2})$. First, let $p \geq 0$ be arbitrary and consider the actual representation sphere $S^{p\sigma_1}$. This representation sphere has an explicit G-CW structure consisting of two equivariant 0-cells indexed by G/G, and a single equivariant k-cell indexed by G/H_1 for each $1 \leq k \leq p$. Then, similar to the proof of Proposition 3.4, the (reduced) cellular chain complex computing the homology $\underline{H}_*(S^{p\sigma_1}; \underline{\mathbb{F}_2})(G/e)$ is given by

$$\mathbb{F}_2[G/G] \xleftarrow{\nabla} \mathbb{F}_2[G/H_1] \xleftarrow{1+t_2} \mathbb{F}_2[G/H_1] \xleftarrow{1+t_2} \cdots \xleftarrow{1+t_2} \mathbb{F}_2[G/H_1].$$

Note that we have written these differentials in terms of the representative t_2 of the non-trivial coset in G/H_1, but we could have instead chosen the representative t_3 so that the differentials can be written as multiplication by $1 + t_3$. By taking H_1, H_2, H_3 and G-fixed points, we obtain a (singly-graded) chain complex of Mackey functors whose homology is precisely $\underline{\pi}_*(S^{p\sigma_1} \wedge H\underline{\mathbb{F}_2})$. The G/e, G/H_1

and G/G levels of this chain complex of Mackey functors is given by

$$\begin{array}{ccccccccc}
\mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \cdots & \xleftarrow{0} & \mathbb{F}_2 \\
1 \downarrow \uparrow 0 & & \Delta \downarrow \uparrow \nabla & & \Delta \downarrow \uparrow \nabla & & & & \Delta \downarrow \uparrow \nabla \\
\mathbb{F}_2[G/G] & \xleftarrow{\nabla} & \mathbb{F}_2[G/H_1] & \xleftarrow{1+t_2} & \mathbb{F}_2[G/H_1] & \xleftarrow{1+t_2} & \cdots & \xleftarrow{1+t_2} & \mathbb{F}_2[G/H_1] \\
1 \downarrow \uparrow 0 & & 1 \downarrow \uparrow 0 & & 1 \downarrow \uparrow 0 & & & & 1 \downarrow \uparrow 0 \\
\mathbb{F}_2[G/G] & \xleftarrow{\nabla} & \mathbb{F}_2[G/H_1] & \xleftarrow{1+t_2} & \mathbb{F}_2[G/H_1] & \xleftarrow{1+t_2} & \cdots & \xleftarrow{1+t_2} & \mathbb{F}_2[G/H_1].
\end{array}$$

Looking instead at the G/H_2 level (as well as again the G/e and G/G levels), we have

$$\begin{array}{ccccccccc}
\mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \cdots & \xleftarrow{0} & \mathbb{F}_2 \\
1 \downarrow \uparrow 0 & & 1 \downarrow \uparrow 0 & & 1 \downarrow \uparrow 0 & & & & 1 \downarrow \uparrow 0 \\
\mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \mathbb{F}_2 & \xleftarrow{0} & \cdots & \xleftarrow{0} & \mathbb{F}_2 \\
1 \downarrow \uparrow 0 & & \Delta \downarrow \uparrow \nabla & & \Delta \downarrow \uparrow \nabla & & & & \Delta \downarrow \uparrow \nabla \\
\mathbb{F}_2[G/G] & \xleftarrow{\nabla} & \mathbb{F}_2[G/H_1] & \xleftarrow{1+t_2} & \mathbb{F}_2[G/H_1] & \xleftarrow{1+t_2} & \cdots & \xleftarrow{1+t_2} & \mathbb{F}_2[G/H_1].
\end{array}$$

Note that the chain complex looking at the G/e, G/H_3 and G/G levels is identical to the above. Computing homology, we therefore see that

$$\underline{\pi}_k(S^{p\sigma_1} \wedge H\underline{\mathbb{F}_2}) = \begin{array}{ccc} & \mathbb{F}_2 & \\ & \downarrow & \searrow \\ 0 & \mathbb{F}_2 & \quad \mathbb{F}_2 \\ & & \\ & 0 & \end{array}$$

for each $0 \leq k \leq p-1$, and $\underline{\pi}_p(S^{p\sigma_1} \wedge H\underline{\mathbb{F}_2})$ is the constant Mackey functor $\underline{\mathbb{F}_2}$. Now, if $p > 0$ then as in Chapter 3 in order to compute $\underline{\pi}_*(S^{-p\sigma_1} \wedge H\underline{\mathbb{F}_2})$ we dualise the reduced cellular chain complex computing $\underline{H}_*(S^{p\sigma_1}; \underline{\mathbb{F}_2})(G/e)$ and then take fixed points under the various subgroups of G to give us a dual chain complex of Mackey functors. The G/e, G/H_1 and G/G levels of this chain complex is given by

$$
\begin{array}{ccccccccc}
\mathbb{F}_2 & \xrightarrow{1} & \mathbb{F}_2 & \xrightarrow{0} & \mathbb{F}_2 & \xrightarrow{0} \cdots \xrightarrow{0} & \mathbb{F}_2 \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} \\
\mathbb{F}_2[G/G] & \xrightarrow{\Delta} & \mathbb{F}_2[G/H_1] & \xrightarrow{1+t_2} & \mathbb{F}_2[G/H_1] & \xrightarrow{1+t_2} \cdots \xrightarrow{1+t_2} & \mathbb{F}_2[G/H_1] \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} \\
\mathbb{F}_2[G/G] & \xrightarrow{\Delta} & \mathbb{F}_2[G/H_1] & \xrightarrow{1+t_2} & \mathbb{F}_2[G/H_1] & \xrightarrow{1+t_2} \cdots \xrightarrow{1+t_2} & \mathbb{F}_2[G/H_1],
\end{array}
$$

and looking at either the G/H_2 or G/H_3 levels (as well as the G/e and G/G levels) we have

$$
\begin{array}{ccccccccc}
\mathbb{F}_2 & \xrightarrow{1} & \mathbb{F}_2 & \xrightarrow{0} & \mathbb{F}_2 & \xrightarrow{0} \cdots \xrightarrow{0} & \mathbb{F}_2 \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} \\
\mathbb{F}_2 & \xrightarrow{1} & \mathbb{F}_2 & \xrightarrow{0} & \mathbb{F}_2 & \xrightarrow{0} \cdots \xrightarrow{0} & \mathbb{F}_2 \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} \\
\mathbb{F}_2[G/G] & \xrightarrow{\Delta} & \mathbb{F}_2[G/H_1] & \xrightarrow{1+t_2} & \mathbb{F}_2[G/H_1] & \xrightarrow{1+t_2} \cdots \xrightarrow{1+t_2} & \mathbb{F}_2[G/H_1].
\end{array}
$$

Taking homology (and assuming for the moment that $p > 1$), we therefore see that the non-zero homotopy Mackey functors are given by

$$
\underline{\pi}_k(S^{-p\sigma_1} \wedge H\underline{\mathbb{F}_2}) \;=\; \begin{array}{ccccc}
 & & \mathbb{F}_2 & & \\
 & & \downarrow & \searrow & \\
0 & & \mathbb{F}_2 & & \mathbb{F}_2 \\
 & & & & \\
 & & 0 & &
\end{array}
$$

for each $-p+1 \leq k \leq -2$ and

$$
\underline{\pi}_{-p}(S^{-p\sigma_1} \wedge H\underline{\mathbb{F}_2}) \;=\; \begin{array}{ccccc}
 & & \mathbb{F}_2 & & \\
 & \nearrow & \downarrow & \searrow & \\
\mathbb{F}_2 & & \mathbb{F}_2 & & \mathbb{F}_2 \\
 & \searrow & \uparrow & \nearrow & \\
 & & \mathbb{F}_2. & &
\end{array}
$$

If $p = 1$, then the only non-zero homotopy Mackey functor is

$$\underline{\pi}_{-1}(S^{-\sigma_1} \wedge H\underline{\mathbb{F}_2}) = \begin{array}{ccc} & 0 & \\ \mathbb{F}_2 & 0 & 0 \\ & \searrow & \\ & \mathbb{F}_2. & \end{array}$$

Notice in particular for every $p \in \mathbb{Z}$ we have that

$$\pi_k^G(S^{p\sigma_1} \wedge H\underline{\mathbb{F}_2}) \cong \pi_k^{C_2}(S^{p\sigma} \wedge H\underline{\mathbb{F}_2}),$$

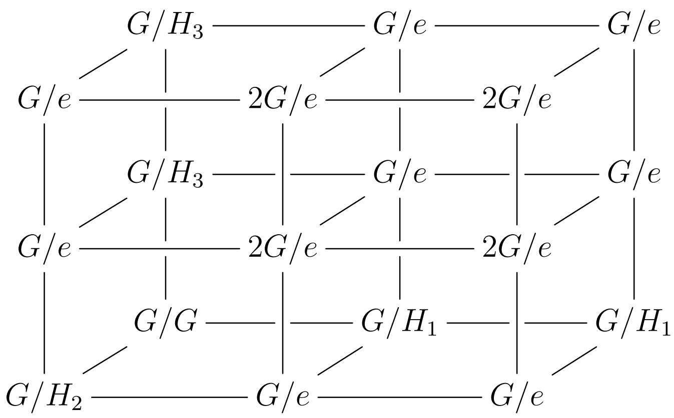
where on the right-hand side we are looking at the C_2-Mackey functor homotopy of $H\underline{\mathbb{F}_2}$ where $\underline{\mathbb{F}_2}$ is the constant C_2-Mackey functor. That is, by looking at $\pi_*^G(S^{p\sigma_1} \wedge H\underline{\mathbb{F}_2})$ we have a copy of the calculation of $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$ from Chapter 3. Moreover, using the notation from Definition 2.21 we see that the restricted H_1-Mackey functor $\downarrow_{H_1}^G \underline{\pi}_k(S^{p\sigma_1} \wedge H\underline{\mathbb{F}_2})$ is precisely the C_2-Mackey functor $\underline{\pi}_k(S^p \wedge H\underline{\mathbb{F}_2})$, and both $\downarrow_{H_2}^G \underline{\pi}_k(S^{p\sigma_1} \wedge H\underline{\mathbb{F}_2})$ and $\downarrow_{H_3}^G \underline{\pi}_k(S^{p\sigma_1} \wedge H\underline{\mathbb{F}_2})$ are equal to the C_2-Mackey functor $\underline{\pi}_k(S^{p\sigma} \wedge H\underline{\mathbb{F}_2})$.

This phenomenon holds more generally due to the relation between the irreducible representations of $C_2 \times C_2$ and C_2 discussed in Section 4.1, whereby

$$\downarrow_{H_1}^G \underline{\pi}_k(S^{p\sigma_1 + q\sigma_2 + r\sigma_3} \wedge H\underline{\mathbb{F}_2}) = \underline{\pi}_k(S^{p+(q+r)\sigma} \wedge H\underline{\mathbb{F}_2}),$$

and similarly we are left with the C_2-Mackey functors $\underline{\pi}_k(S^{q+(p+r)\sigma} \wedge H\underline{\mathbb{F}_2})$ and $\underline{\pi}_k(S^{r+(p+q)\sigma} \wedge H\underline{\mathbb{F}_2})$ if we restrict to the C_2-subgroups H_2 and H_3 respectively.

Note that the above discussion about $\underline{\pi}_k(\Sigma^{p\sigma_1} H\underline{\mathbb{F}_2})$ is symmetric if we want to instead look at either $\underline{\pi}_k(\Sigma^{q\sigma_2} H\underline{\mathbb{F}_2})$ or $\underline{\pi}_k(\Sigma^{r\sigma_3} H\underline{\mathbb{F}_2})$. Now, given $(p, q, r) \in \mathbb{Z}^3$, our trigraded complex of Mackey functors computing $\underline{\pi}_*(\Sigma^{p\sigma_1 + q\sigma_2 + r\sigma_3} H\underline{\mathbb{F}_2})$ will be obtained by taking the tensor product of the G/e levels of the above singly graded chain complexes of Mackey functors computing $\underline{\pi}_*(\Sigma^{p\sigma_1} H\underline{\mathbb{F}_2})$, $\underline{\pi}_*(\Sigma^{q\sigma_2} H\underline{\mathbb{F}_2})$ and $\underline{\pi}_*(\Sigma^{r\sigma_3} H\underline{\mathbb{F}_2})$ and then taking fixed points. For $p, q, r \geq 0$, this resulting triple complex can be viewed as the reduced cellular chain complex corresponding to the product G-CW structure on $S^{p\sigma_1 + q\sigma_2 + r\sigma_3} = S^{p\sigma_1} \wedge S^{q\sigma_2} \wedge S^{r\sigma_3}$ with respect to our above G-CW structures on the actual representation spheres $S^{p\sigma_1}$, $S^{q\sigma_2}$ and $S^{r\sigma_3}$. Since we are looking at the reduced cellular chain complexes, we may ignore the second 0-cells in each of these three G-CW structures and thus we can visualise the product G-CW structure on $S^{p\sigma_1+q\sigma_2+r\sigma_3}$ as follows:



Note we are using here that, as G-sets,

$$G/H_i \times G/H_j \cong G/e$$

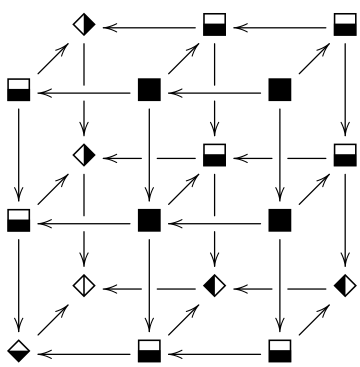
for each $i, j \in \{1, 2, 3\}$ with $i \neq j$, and that

$$G/H_1 \times G/H_2 \times G/H_3 \cong G/e \sqcup G/e.$$

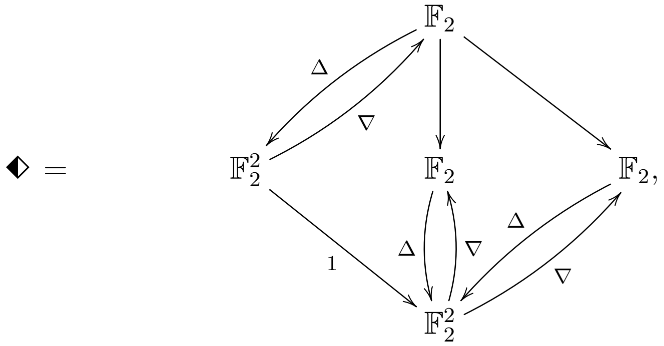
Furthermore, notice that

$$G/H_i \times G/H_i \cong G/H_i \sqcup G/H_i$$

for each $i \in \{1, 2, 3\}$ as in Chapter 3, thinking of G/H_i as a copy of C_2/e. Now, each Mackey functor in our triple complex is the fixed-point Mackey functor corresponding to one of the $\mathbb{F}_2[G]$-modules $\mathbb{F}_2[G/G]$, $\mathbb{F}_2[G/H_1]$, $\mathbb{F}_2[G/H_2]$, $\mathbb{F}_2[G/H_3]$, $\mathbb{F}_2[G/e]$ or $\mathbb{F}_2[2G/e]$. For $p, q, r \geq 0$ our triple complex of Mackey functors looks as follows (in particular when $p = r = 2$ and $q = 1$). The fixed-point Mackey functors that these symbols represent are described explicitly below.

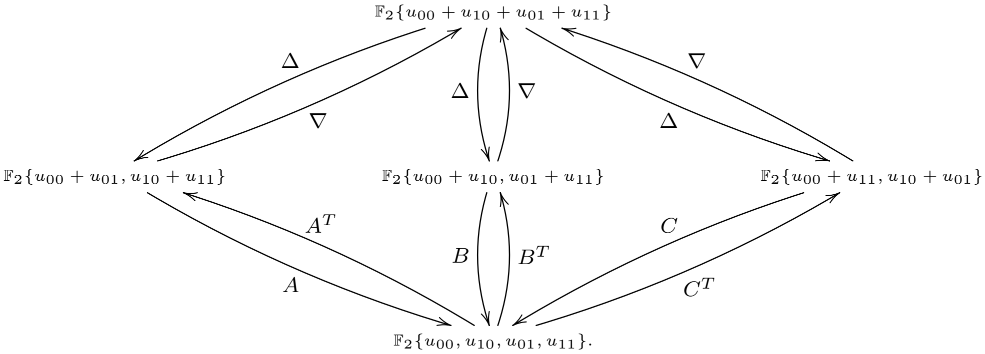


We know from before that ◈ is the constant Mackey functor $\underline{\mathbb{F}_2}$. Furthermore, we saw that the fixed-point Mackey functor corresponding to $\mathbb{F}_2[G/H_1]$ is given by



and the fixed-point Mackey functors ⬙ and ⬗ corresponding to $\mathbb{F}_2[G/H_2]$ and $\mathbb{F}_2[G/H_3]$ are symmetric where the two copies of $\mathbb{F}_2^2$ now appear at the G/e and G/H_2 or the G/e and G/H_3 levels respectively. The fixed-point Mackey functors corresponding to the $\mathbb{F}_2[G]$-modules $\mathbb{F}_2[G/e]$ and $\mathbb{F}_2[2G/e]$ are more complicated. First, note that our trigraded complex of Mackey functors includes the fixed-point Mackey functor corresponding to the $\mathbb{F}_2[G]$-module $\mathbb{F}_2[G/e]$ in the form of $\mathbb{F}_2[G/H_1 \times G/H_2]$, $\mathbb{F}_2[G/H_1 \times G/H_3]$ and $\mathbb{F}_2[G/H_2 \times G/H_3]$. However, as in the above diagram of our triple complex we have named each of these three Mackey functors ⬓, and the reason for this is that we can choose bases for the three $\mathbb{F}_2[G]$-modules such that the three resulting fixed-point Mackey functors are equal.

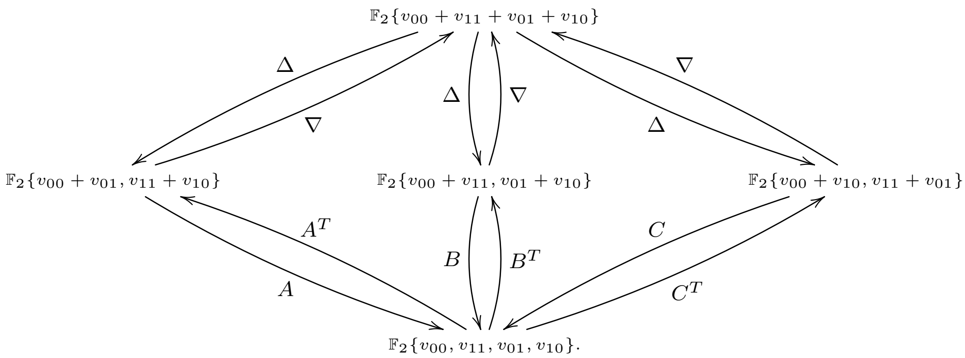
We first look at the fixed-point Mackey functor associated to $\mathbb{F}_2[G/H_1 \times G/H_2]$. Similar to the proof of Theorem 3.10, for ease of notation we will write elements in $\mathbb{F}_2[G/H_1 \times G/H_2]$ as $u_{ij} = (a_i, b_j)$ for $0 \leq i, j \leq 1$, where $a_0 = H_1$, $a_1 = t_2H_1$, $b_0 = H_2$ and $b_1 = t_1H_2$. Then, we use the ordered basis $\{u_{00}, u_{10}, u_{01}, u_{11}\}$ for $\mathbb{F}_2[G/H_1 \times G/H_2]$. In this basis, the corresponding fixed-point Mackey functor is given by



Here the matrices A, B and C are given by

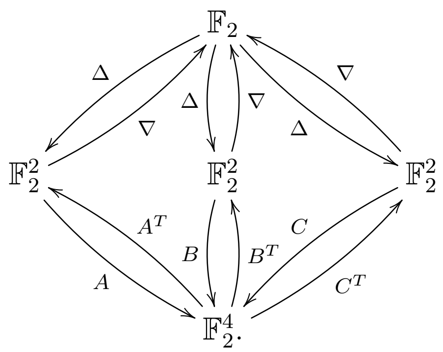
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Similarly, if we write elements in $\mathbb{F}_2[G/H_1 \times G/H_3]$ as $v_{ij} = (a_i, c_k)$ for $0 \leq i, k \leq 1$ where $c_0 = H_3$ and $c_1 = t_1H_3 = t_2H_3$ and use the basis $\{v_{00}, v_{11}, v_{01}, v_{10}\}$ for $\mathbb{F}_2[G/H_1 \times G/H_3]$, then the corresponding fixed-point Mackey functor is given by



The final case looking at $\mathbb{F}_2[G/H_2 \times G/H_3]$ is similar. Therefore, in our trigraded complex of Mackey functors we can view each of these three fixed-point Mackey functors corresponding to $\mathbb{F}_2[G]$-modules isomorphic to $\mathbb{F}_2[G/e]$ as

⬓ =



Finally, the fixed-point Mackey functor ■ corresponding to the $\mathbb{F}_2[G]$-module $\mathbb{F}_2[2G/e] = \mathbb{F}_2[G/H_1 \times G/H_2 \times G/H_3]$ can be viewed as the direct sum of two copies of the above Mackey functor ⬓ whereby we square each $\mathbb{F}_2$-vector space on the various levels of the Mackey functor, and each transfer or restriction map F in ⬓ becomes the transfer or restriction map given by the block matrix

$$\begin{bmatrix} F & 0 \\ 0 & F \end{bmatrix}$$

in the Mackey functor ■. By viewing $\Delta\colon \mathbb{F}_2 \to \mathbb{F}_2^2$ as the matrix $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and $\nabla\colon \mathbb{F}_2^2 \to \mathbb{F}_2$ as the matrix $\left[\begin{smallmatrix}1 & 1\end{smallmatrix}\right]$, note that

$$B = \begin{bmatrix}\Delta & 0\\ 0 & \Delta\end{bmatrix} \text{ and } B^T = \begin{bmatrix}\nabla & 0\\ 0 & \nabla\end{bmatrix}.$$

To obtain these transfer and restriction maps, we are writing elements in $\mathbb{F}_2[G/H_1 \times G/H_2 \times G/H_3]$ as $z_{ijk} = (a_i, b_j, c_k)$ for $0 \leq i, j, k \leq 1$ and using the basis

$$\{z_{000}, z_{101}, z_{011}, z_{110}, z_{111}, z_{010}, z_{100}, z_{001}\}$$

of $\mathbb{F}_2[2G/e]$, which gives us the bases

$$\{z_{000} + z_{011}, z_{101} + z_{110}, z_{111} + z_{100}, z_{010} + z_{001}\},$$

$$\{z_{000} + z_{101}, z_{011} + z_{110}, z_{111} + z_{010}, z_{100} + z_{001}\},$$

$$\{z_{000} + z_{110}, z_{101} + z_{011}, z_{111} + z_{001}, z_{010} + z_{100}\},$$

$$\{z_{000} + z_{101} + z_{011} + z_{110}, z_{111} + z_{010} + z_{100} + z_{001}\}$$

of $\mathbb{F}_2[2G/e]^{H_1}$, $\mathbb{F}_2[2G/e]^{H_2}$, $\mathbb{F}_2[2G/e]^{H_3}$ and $\mathbb{F}_2[2G/e]^{G}$ respectively. We now need to understand the differentials in our triple complex of Mackey functors. Let d^1, d^2 and d^3 denote the differentials for the bottom levels of the singly-graded chain complexes of fixed-point Mackey functors computing $\underline{\pi}_*(\Sigma^{p\sigma_1} H\underline{\mathbb{F}_2})$, $\underline{\pi}_*(\Sigma^{q\sigma_2} H\underline{\mathbb{F}_2})$ and $\underline{\pi}_*(\Sigma^{r\sigma_3} H\underline{\mathbb{F}_2})$ respectively. Then, the differential for the bottom level of our triple complex of fixed-point Mackey functors with homology $\underline{\pi}_*(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{F}_2})$ is given by

$$d = d^1 + d^2 + d^3,$$

noting for example that when we apply d^1 to an element of the bottom level of a Mackey functor in our triple complex it acts as the identity on each component of the tensor product that does not come from the singly graded chain complex of Mackey functors computing $\underline{\pi}_*(\Sigma^{p\sigma_1} H\underline{\mathbb{F}_2})$. The differentials at higher levels of our Mackey functors are given by applying the differentials d^1, d^2 and d^3 at the bottom level and then taking fixed points.

The fixed points of these three differentials d^1, d^2 and d^3 can be computed explicitly using our above bases in a similar manner to the proof of Theorem 3.10. For example, to compute the differential ■ $\to$ ■ corresponding to d^1 given by

$(1+t_2)\times \mathrm{id}\times \mathrm{id}$, we first observe that at the bottom level we have that

$$\begin{aligned} z_{000} &\mapsto z_{000}+z_{100}, & z_{111} &\mapsto z_{111}+z_{011},\\ z_{101} &\mapsto z_{101}+z_{001}, & z_{010} &\mapsto z_{010}+z_{110},\\ z_{011} &\mapsto z_{011}+z_{111}, & z_{100} &\mapsto z_{100}+z_{000},\\ z_{110} &\mapsto z_{110}+z_{010}, & z_{001} &\mapsto z_{001}+z_{101}. \end{aligned}$$

If we take H_1-fixed points, i.e. extend this map linearly to the basis elements of $\mathbb{F}_2[G/e]^{H_1}$, we see that

$$\begin{aligned} z_{000}+z_{011} &\mapsto z_{000}+z_{011}+z_{111}+z_{100}, & z_{111}+z_{100} &\mapsto z_{000}+z_{011}+z_{111}+z_{100}\\ z_{101}+z_{110} &\mapsto z_{101}+z_{110}+z_{010}+z_{001}, & z_{010}+z_{001} &\mapsto z_{101}+z_{110}+z_{010}+z_{001}, \end{aligned}$$

and hence we are left with the 4×4 block matrix $\begin{bmatrix} A^T \\ A^T \end{bmatrix}$. If we instead take H_2 or H_3-fixed points, then in either case we get the matrix $\begin{bmatrix} C^T \\ C^T \end{bmatrix}$. If we take G-fixed points, then we get the matrix $\begin{bmatrix} \triangledown \\ \triangledown \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. In general, consider the chain complex of Mackey functors

$$⬓ \leftarrow \blacksquare \leftarrow \blacksquare \leftarrow \blacksquare \leftarrow \cdots$$

associated to d^i for some fixed $i\in\{1,2,3\}$. Then, the G/H_i level of this chain complex of Mackey functors is given by

$$\mathbb{F}_2^2 \xleftarrow{A^T} \mathbb{F}_2^4 \xleftarrow{\begin{bmatrix} A^T \\ A^T \end{bmatrix}} \mathbb{F}_2^4 \xleftarrow{\begin{bmatrix} A^T \\ A^T \end{bmatrix}} \cdots$$

and the G/H_j level for each $j\neq i$ is given by

$$\mathbb{F}_2^2 \xleftarrow{C^T} \mathbb{F}_2^4 \xleftarrow{\begin{bmatrix} C^T \\ C^T \end{bmatrix}} \mathbb{F}_2^4 \xleftarrow{\begin{bmatrix} C^T \\ C^T \end{bmatrix}} \cdots$$

Furthermore, the G/G level of this chain complex of Mackey functors is given by

$$\mathbb{F}_2 \xleftarrow{\nabla} \mathbb{F}_2^2 \xleftarrow{\begin{bmatrix} \triangledown \\ \triangledown \end{bmatrix}} \mathbb{F}_2^2 \xleftarrow{\begin{bmatrix} \triangledown \\ \triangledown \end{bmatrix}} \cdots$$

Note that we are not explicitly giving the differentials at the bottom level of our chain complex of Mackey functors as we already know the bottom and middle levels of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ as well as the transfer and restriction maps between them from the C_2-equivariant calculation in Chapter 3, as discussed earlier in this section.

Next, consider the chain complex of Mackey functors

$$⬘ \leftarrow ⬓ \leftarrow ⬓ \leftarrow ⬓ \leftarrow \cdots$$

associated to d^1. Then, using our bases for the various levels of these Mackey functors (in particular we are looking at our earlier basis for $\mathbb{F}_2[G/H_1 \times G/H_2]$), we see that both the G/H_1 and G/H_3 levels of this chain complex of Mackey functors are given by

$$\mathbb{F}_2 \xleftarrow{\nabla} \mathbb{F}_2^2 \xleftarrow{\begin{bmatrix}\triangledown & \\ & \triangledown\end{bmatrix}} \mathbb{F}_2^2 \xleftarrow{\begin{bmatrix}\triangledown & \\ & \triangledown\end{bmatrix}} \cdots$$

and the G/H_2 level is given by

$$\mathbb{F}_2^2 \xleftarrow{0} \mathbb{F}_2^2 \xleftarrow{0} \mathbb{F}_2^2 \xleftarrow{0} \cdots$$

Finally, we see that the G/G level is given by the chain complex

$$\mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \xleftarrow{0} \cdots$$

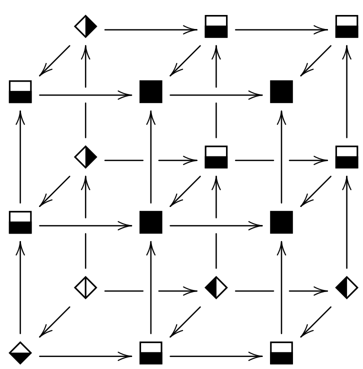
The result is symmetric for the various other singly-graded chain complexes contained in our trigraded complex of Mackey functors of the form

⬖ ← ⬓ ← ⬓ ← ⬓ ← ⋯

or

⬗ ← ⬓ ← ⬓ ← ⬓ ← ⋯

Note that so far we have been discussing the trigraded complex of Mackey functors computing $\underline{\pi}_*(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{F}_2})$ for $p, q, r \geq 0$. However, if at least one of p, q or r is negative then the corresponding trigraded complex of Mackey functors will have reversed arrows in at least one of the three directions compared to our trigraded complex for actual representations, which comes from the fact that we will be taking the tensor product with one of the dual singly-graded chain complexes of Mackey functors constructed earlier in this section. For example, if we are looking at $\underline{\pi}_*(\Sigma^{p\sigma_1-q\sigma_2-r\sigma_3} H\underline{\mathbb{F}_2})$ where $p \geq 0$ and $q, r \geq 1$, then our trigraded complex of Mackey functors now looks as follows:



The arrows in one of these trigraded complexes of Mackey functors that are reversed when compared to the trigraded complex for actual representations can be computed similarly using the same bases at each level of these fixed-point Mackey functors from before, and using our earlier description of the differentials at the bottom level of one of the dual singly-graded complexes computing either $\underline{\pi}_*(\Sigma^{-p\sigma_1} H\underline{\mathbb{F}_2})$, $\underline{\pi}_*(\Sigma^{-q\sigma_2} H\underline{\mathbb{F}_2})$ or $\underline{\pi}_*(\Sigma^{-r\sigma_3} H\underline{\mathbb{F}_2})$ where $p, q, r \geq 1$. In particular, suppose that our trigraded complex of Mackey functors contains the dual chain complex

$$⬓ \to \blacksquare \to \blacksquare \to \blacksquare \to \cdots$$

associated to d^i for some fixed $i \in \{1, 2, 3\}$. Then similar to before, the G/H_i level of this chain complex of Mackey functors is given by

$$\mathbb{F}_2^2 \xrightarrow{A} \mathbb{F}_2^4 \xrightarrow{\begin{bmatrix} A & A \end{bmatrix}} \mathbb{F}_2^4 \xrightarrow{\begin{bmatrix} A & A \end{bmatrix}} \cdots$$

and the G/H_j level for each $j \neq i$ is given by

$$\mathbb{F}_2^2 \xrightarrow{C} \mathbb{F}_2^4 \xrightarrow{\begin{bmatrix} C & C \end{bmatrix}} \mathbb{F}_2^4 \xrightarrow{\begin{bmatrix} C & C \end{bmatrix}} \cdots$$

The G/G level of this chain complex of Mackey functors is given by

$$\mathbb{F}_2 \xrightarrow{\Delta} \mathbb{F}_2^2 \xrightarrow{\begin{bmatrix} \Delta & \Delta \end{bmatrix}} \mathbb{F}_2^2 \xrightarrow{\begin{bmatrix} \Delta & \Delta \end{bmatrix}} \cdots$$

Next, consider the dual chain complex

$$⬙ \to ⬓ \to ⬓ \to ⬓ \to \cdots$$

associated to d^1, so in particular we view ⬓ as being the fixed-point Mackey functor associated to the $\mathbb{F}_2[G]$-module $\mathbb{F}_2[G/H_1 \times G/H_2]$. The G/H_1 and G/H_3 levels of this chain complex of Mackey functors are given by

$$\mathbb{F}_2 \xrightarrow{\Delta} \mathbb{F}_2^2 \xrightarrow{\begin{bmatrix} \Delta & \Delta \end{bmatrix}} \mathbb{F}_2^2 \xrightarrow{\begin{bmatrix} \Delta & \Delta \end{bmatrix}} \cdots$$

and the G/H_2 level is given by

$$\mathbb{F}_2^2 \xrightarrow{1} \mathbb{F}_2^2 \xrightarrow{0} \mathbb{F}_2^2 \xrightarrow{0} \cdots$$

Similarly, we see that the G/G level is given by

$$\mathbb{F}_2 \xrightarrow{1} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \cdots$$

We have now constructed for each $(p, q, r) \in \mathbb{Z}^3$ a trigraded complex of fixed-point Mackey functors whose homology is precisely $\underline{\pi}_*(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$. However, even though we can ignore the bottom levels of these Mackey functors as we already know the G/e and G/H_i levels and the transfer and restriction maps between them in $\underline{\pi}_*(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$, computing the homology of this triple complex (with respect to the total differential) in general is difficult due to the large powers of $\mathbb{F}_2$ in the total complex at the top and middle levels. The remainder of this chapter discusses ways that we can overcome this issue of computing the homology of this large chain complex mainly through the lens of the multiplicative structure of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$.

4.3 The Poincaré series of Holler-Kriz and duality

The additive structure of the top level $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ is computed in [2], and the authors present the result in the form of Poincaré series encoding the dimensions of the $\mathbb{F}_2$-vector spaces appearing in each degree. From the perspective of Section 4.2, these Poincaré series can be obtained by computing the homology of the top level of the trigraded complex of Mackey functors associated to a given $(p, q, r) \in \mathbb{Z}^3$. When we focus at a single level of our trigraded complex, the homology can be computed by iteratively running the spectral sequence of a double complex.

Recall that in general if $C = C_{*,*}$ is a double complex, then we have two homological spectral sequences

$$E^2_{s,t} = H_s(H_t(C, d^v), d^h) \Rightarrow H_{s+t}(C, d^v + d^h)$$

and

$$E^2_{s,t} = H_s(H_t(C, d^h), d^v) \Rightarrow H_{s+t}(C, d^v + d^h)$$

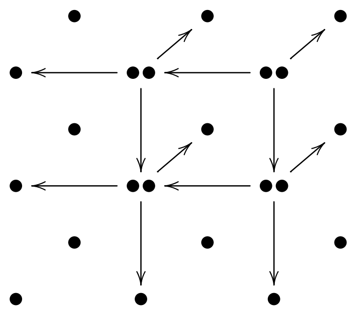
converging to the homology of C with respect to the total differential $d = d^v + d^h$, where the horizontal differential d^h decreases the first grading by one and the vertical differential d^v decreases the second grading by one. However, when discussing $\pi^G_\bigstar H\underline{\mathbb{F}_2}$ we are dealing with complexes with three gradings, so as mentioned above we will need to use the spectral sequence of a double complex iteratively. That is, we first run the spectral sequence for the double complex given by setting one of the three gradings to be zero, and then we run the spectral sequence again for the double complex which in one direction is given by the third grading we set before to be zero and in the other direction is the total complex of the previous double complex. We will show explicitly how this is done in the case of computing $\pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$ for $p, q, r \geq 0$.

Theorem 4.2 (Holler-Kriz). *Suppose that $p, q, r \geq 0$. Then, the Poincaré series for $\pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$ is given by*

$$(1+\cdots+x^p)(1+\cdots+x^q)(1+\cdots+x^r)-x^2(1+\cdots+x^{p-1})(1+\cdots+x^{q-1})(1+\cdots+x^{r-1}).$$

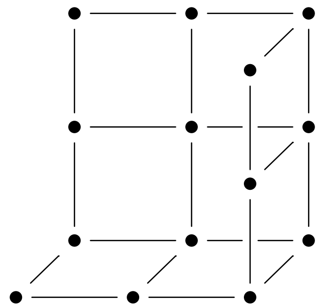
Remark 4.3. In the above Poincaré series, the coefficient of x^i is the dimension of the $\mathbb{F}_2$-vector space at degree i. Furthermore, we have re-written the form of the Poincaré series given in [2] to make it symmetric in p, q and r.

Proof. As discussed in Section 4.2, the top level of the trigraded complex of Mackey functors computing $\underline{\pi}_*(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$ is given by the triple complex



Note that each $\bullet$ represents a copy of $\mathbb{F}_2$ and each $\bullet\bullet$ represents a copy of $\mathbb{F}_2^2$, and the differentials were given in Section 4.2. Now, we want to compute the homology of this triple complex with respect to the total differential $d = d^1 + d^2 + d^3$. Using the spectral sequence of a double complex, there are six ways to compute the homology $\pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$. For example, we can first take homology with respect to d^1 and run the spectral sequence of a double complex converging to the homology with respect to $d^1 + d^2$, and then take homology with respect to d^3 and run the spectral sequence of a double complex converging to the homology with respect to $d^1 + d^2 + d^3 = d$. The other five ways of iteratively using the spectral sequence of a double complex correspond to the other five permutations of d^1, d^2 and d^3.

To compute the homology $\pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$, we will first run the spectral sequence converging to the homology with respect to $d^1 + d^2$ where we first compute homology with respect to d^1, recalling from our above diagram of the triple complex the differential d^1 points to the left. The E^1-page of this spectral sequence then looks as follows:



We see that there are no higher differentials, so the spectral sequence collapses on the E^1-page. Furthermore, there are no non-zero differentials in the σ_3-direction, so the above diagram tells us the complete homology $\pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$ geometrically as three adjacent faces of a cube with side lengths $p+1$, $q+1$ and $r+1$. More precisely, each point (with integer coordinates) inside this cube gives us a term in the Poincaré series, so if we were looking at the whole cube then we would have the Poincaré series

$$(1+\cdots+x^p)(1+\cdots+x^q)(1+\cdots+x^r).$$

However, looking at our above diagram we need to remove the cube represented by

$$x^2(1+\cdots+x^{p-1})(1+\cdots+x^{q-1})(1+\cdots+x^{r-1})$$

sitting inside it, so we have that the Poincare series for $\pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$ is given by

$$(1+\cdots+x^p)(1+\cdots+x^q)(1+\cdots+x^r)-x^2(1+\cdots+x^{p-1})(1+\cdots+x^{q-1})(1+\cdots+x^{r-1}).$$

□

Note that the Poincaré series given in Theorem 4.2 is symmetric in p, q and r as one would expect, but it can also be re-written in a more concise form as

$$(1+\cdots+x^p)(1+\cdots+x^q)+(1+\cdots+x^{p+q})(x+\cdots+x^r),$$

although this polynomial is no longer symmetric in p, q and r. This form of the Poincaré series can be obtained by instead looking at the horizontal cross sections of our above diagram of the E^1-page. If we want to compute $\pi_*^G(\Sigma^V H\underline{\mathbb{F}_2})$ for virtual (or non-actual) representations $V = p\sigma_1 + q\sigma_2 + r\sigma_3$, then our spectral sequences do not generally collapse on the E^1-page as in the proof of Theorem 4.2. For convenience, we list the Poincaré series for virtual representations below as presented in [3, Section 2.6].

Theorem 4.4 (Holler-Kriz). *Suppose that $p, q, r \geq 1$. Then, the Poincaré series for $\pi_*^G(\Sigma^{-p\sigma_1-q\sigma_2-r\sigma_3}H\underline{\mathbb{F}_2})$ is given by*

$$\frac{1}{x^{p+q+r}}[(1+x+\cdots+x^{q+r-2})(1+\cdots+x^{p-2})+x^{p-1}(1+\cdots+x^{r-1})(1+\cdots+x^{q-1})].$$

Theorem 4.5 (Holler-Kriz). *Suppose that $p, q \geq 0$ and $r \geq 1$. Then, the Poincaré series for $\pi_*^G(\Sigma^{p\sigma_1+q\sigma_2-r\sigma_3}H\underline{\mathbb{F}_2})$ is given by*

$$\left(\frac{1}{x^r}+\cdots+\frac{1}{x}\right)(1+x+\cdots+x^{r-2})+x^r(1+\cdots+x^{p-r})(1+\cdots+x^{q-r}).$$

Theorem 4.6 (Holler-Kriz). *Suppose that $p \geq 0$ and $q, r \geq 1$. Then, the Poincaré series for $\pi_*^G(\Sigma^{p\sigma_1-q\sigma_2-r\sigma_3}H\underline{\mathbb{F}_2})$ is given by*

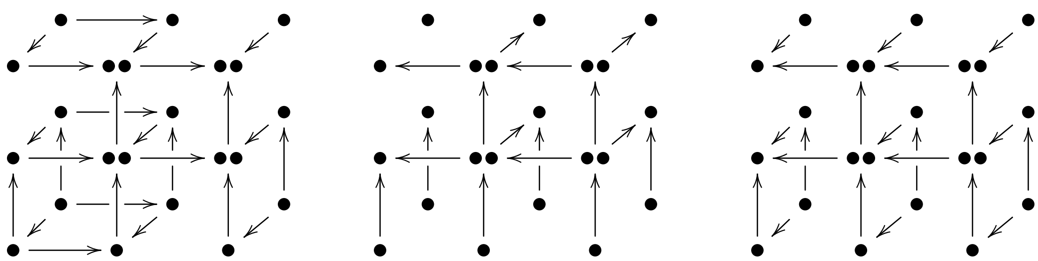
$$\frac{1}{x^{q+r-p}}(1+\cdots+x^{q-p-2})(1+\cdots+x^{r-p-2})+\frac{1}{x^{p+1}}(1+\cdots+x^{p})(1+\cdots+x^{p-1})$$

in the case that $q, r \geq p+1$, and is given by

$$\frac{1}{x^q}(1+\cdots+x^{q-2})(1+\cdots+x^{p-r})+\frac{1}{x^r}(1+\cdots+x^{p-1})(1+\cdots+x^{r-1})$$

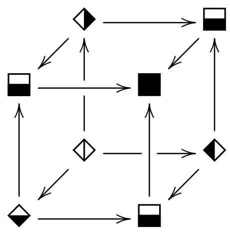
in the case that $p \geq r$, and the case where $p \geq q$ is symmetric.

The relevant triple complexes for Theorems 4.4, 4.5 and 4.6 are given respectively by

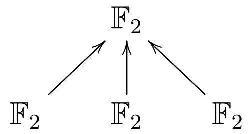


Now, one might hope to use the spectral sequence of a double complex on the level of Mackey functors in order to compute explicitly the complete Mackey functor homotopy $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$. However, when we are working with this spectral sequence of Mackey functors we run in to the problem of exotic transfers and exotic restrictions. That is, when we reach the E^∞-page of our spectral sequence of Mackey functors there may be non-zero transfers or restrictions that are not visible due to them being in higher filtration, although in many cases we can resolve this problem using Proposition 2.15 or that $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ consists of cohomological Mackey functors.

Example 4.7. Suppose that we want to compute the Mackey functor homotopy $\underline{\pi}_*(\Sigma^{-\sigma_1-\sigma_2-\sigma_3}H\underline{\mathbb{F}_2})$. By Theorem 4.4 we know that the top level contains only a copy of $\mathbb{F}_2$ in degree -3. Now, in this case our triple complex of Mackey functors is given by



Note that we can ignore the G/e levels of these Mackey functors and the transfer and restriction maps between the G/e and G/H_i levels in $\underline{\pi}_*(\Sigma^{-\sigma_1-\sigma_2-\sigma_3}H\underline{\mathbb{F}_2})$ since we already know these as discussed in Section 4.2. If we take homology in the σ_1-direction, and then in the σ_2-direction and finally in the σ_3-direction, we are left with the partial Mackey functor



in degree -3 (and zero in every other degree). However, we want to determine whether there are also non-zero restriction maps in this portion of the $(C_2 \times C_2)$-Mackey functor $\underline{\pi}_{-3}(\Sigma^{-\sigma_1-\sigma_2-\sigma_3}H\underline{\mathbb{F}_2})$. If there was a non-zero restriction map $\mathrm{Res}^G_{H_i}$ for some $i \in \{1,2,3\}$, then we must have that $\mathrm{Res}^G_{H_i} \colon \mathbb{F}_2 \to \mathbb{F}_2$ is the identity map. But by Proposition 2.15 we know that

$$\mathrm{Res}^G_{H_i} \circ \mathrm{Tr}^G_{H_i} = \sum_{\gamma \in W_{H_i}(G)} \gamma$$

which is therefore the zero map as the action of the Weyl group $W_{H_i}(G)$ on the $\mathbb{F}_2$ at the G/H_i level is trivial, so we have reached a contradiction. Note that we could have alternatively deduced that there are no restrictions maps using that this Mackey functor is cohomological, and so the composite $\mathrm{Tr}^G_{H_i} \circ \mathrm{Res}^G_{H_i}$ is multiplication by the index $[G : H_i] = 2$, i.e. is the zero map.

Hence, since the restriction of $\underline{\pi}_{-3}(\Sigma^{-\sigma_1-\sigma_2-\sigma_3}H\underline{\mathbb{F}_2})$ to any of the C_2-subgroups H_i is the C_2-Mackey functor $\underline{\pi}_{-2}(\Sigma^{-2\sigma}H\underline{\mathbb{F}_2})$ which we know from Chapter 3, it follows that the only non-zero homotopy Mackey functor in $\underline{\pi}_*(\Sigma^{-\sigma_1-\sigma_2-\sigma_3}H\underline{\mathbb{F}_2})$ is the dual constant Mackey functor $\underline{\mathbb{F}_2^*}$ in degree -3.

Using the calculation of Example 4.7 and Anderson duality, we can now explain as in Chapter 3 how for every $(p,q,r) \in \mathbb{Z}^3$ we have that Theorem 4.4 follows from Theorem 4.2 and the three symmetric versions of Theorem 4.6 follow from the three symmetric versions of Theorem 4.5, and more generally on the level of Mackey functors.

Proposition 4.8 (Guillou-Yarnall)**.** *As $(C_2 \times C_2)$-spectra, we have that $\Sigma^4 H\underline{\mathbb{F}_2} \simeq \Sigma^\rho H\underline{\mathbb{F}_2^*}$.*

Remark 4.9. This twisting is proved in [3, Proposition 4.2], but we present an alternative proof here that is analogous to the proof of Proposition 3.8.

Proof. It suffices to show that $\Sigma^{4-\rho} H\underline{\mathbb{F}_2} \simeq H\underline{\mathbb{F}_2^*}$, and by the uniqueness of Eilenberg-MacLane spectra we just need to show that they have the same homotopy Mackey functors. First, note that

$$\begin{aligned}\underline{\pi}_k(\Sigma^{4-\rho} H\underline{\mathbb{F}_2}) &= \underline{\pi}_k(\Sigma^{3-\sigma_1-\sigma_2-\sigma_3} H\underline{\mathbb{F}_2}) \\ &= \underline{\pi}_{k-3}(\Sigma^{-\sigma_1-\sigma_2-\sigma_3} H\underline{\mathbb{F}_2}).\end{aligned}$$

Now, from Example 4.7 (or alternatively by the result of Theorem 4.35) we know that

$$\underline{\pi}_{k-3}(\Sigma^{-\sigma_1-\sigma_2-\sigma_3} H\underline{\mathbb{F}_2}) = \begin{cases} \underline{\mathbb{F}_2^*} & \text{if } k-3=-3, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, it follows that

$$\underline{\pi}_k(\Sigma^{4-\rho} H\underline{\mathbb{F}_2}) = \begin{cases} \underline{\mathbb{F}_2^*} & \text{if } k=0, \\ 0 & \text{otherwise} \end{cases}$$

which are precisely the homotopy Mackey functors of $H\underline{\mathbb{F}_2^*}$. □

Recall by Proposition 3.9 (applied now to $(C_2 \times C_2)$-Mackey functors) that if $\underline{M}$ is any $\underline{\mathbb{F}_2}$-module, then the Anderson dual $I_{\mathbb{F}_2} H\underline{M}$ of the Eilenberg-MacLane spectrum $H\underline{M}$ is precisely $H\underline{M}^*$. That is, by our characterisation of the Anderson dual of an $H\underline{\mathbb{F}_2}$-module from Section 3.2 we have that

$$\underline{\pi}_V(H\underline{M}^*) \cong (\underline{\pi}_{-V} H\underline{M})^*.$$

Let $k, p, q, r \in \mathbb{Z}$ be arbitrary. Then, by the above discussion we have that

$$\begin{aligned}(\underline{\pi}_k(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{F}_2}))^* &= (\underline{\pi}_{k-p\sigma_1-q\sigma_2-r\sigma_3}(H\underline{\mathbb{F}_2}))^* \\ &\cong \underline{\pi}_{-k+p\sigma_1+q\sigma_2+r\sigma_3}(H\underline{\mathbb{F}_2^*}) \\ &= \underline{\pi}_{-k}(\Sigma^{-p\sigma_1-q\sigma_2-r\sigma_3} H\underline{\mathbb{F}_2^*}).\end{aligned}$$

However, by Proposition 4.8 we have that $\Sigma^{3-\sigma_1-\sigma_2-\sigma_3} H\underline{\mathbb{F}_2} \simeq H\underline{\mathbb{F}_2^*}$. That is,

$$\begin{aligned}\underline{\pi}_{-k}(\Sigma^{-p\sigma_1-q\sigma_2-r\sigma_3} H\underline{\mathbb{F}_2^*}) &\cong \underline{\pi}_{-k}(\Sigma^{3-(p+1)\sigma_1-(q+1)\sigma_2-(r+1)\sigma_3} H\underline{\mathbb{F}_2}) \\ &= \underline{\pi}_{-k-3}(\Sigma^{-(p+1)\sigma_1-(q+1)\sigma_2-(r+1)\sigma_3} H\underline{\mathbb{F}_2}).\end{aligned}$$

Hence, we have shown that

$$\underline{\pi}_k(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{F}_2}) \cong (\underline{\pi}_{-k-3}(\Sigma^{-(p+1)\sigma_1-(q+1)\sigma_2-(r+1)\sigma_3} H\underline{\mathbb{F}_2}))^*.$$

In particular, this explains how we can view Theorems 4.4 and 4.6 as corollaries of Theorems 4.2 and 4.5 respectively, where we only look at the top levels of these homotopy Mackey functors, noting that for every $n \geq 1$ we indeed have that

$$\mathrm{Hom}(\mathbb{F}_2^n, \mathbb{F}_2) \cong \mathbb{F}_2^n,$$

recalling that the top levels of our homotopy Mackey functors are always finite-dimensional $\mathbb{F}_2$-vector spaces.

4.4 The ring structure of the positive cone

In this section we will compute the complete Mackey functor structure of the homotopy $\underline{\pi}_*(\Sigma^V H\underline{\mathbb{F}_2})$ for actual representations V, and we express our answer as a single Mackey functor of $RO(G)$-graded rings which will in fact be an $RO(G)$-graded Green functor. In particular, we derive the ring structure of the top level $\pi_*^G(\Sigma^V H\underline{\mathbb{F}_2})$ for actual representations V with additive structure given previously by Theorem 4.2. First, we introduce some analogous terminology from Chapter 3 that will be used throughout the remainder of this chapter.

Definition 4.10. The *positive cone* is the subring of $\pi_\bigstar^G H\underline{\mathbb{F}_2}$ given by the direct sum of all $\pi_*^G(\Sigma^V H\underline{\mathbb{F}_2})$ for V an actual representation of G, i.e. $V = p\sigma_1+q\sigma_2+r\sigma_3$ with $p, q, r \geq 0$.

That is, using the terminology of Definition 4.10 our goal in this chapter is to compute the complete Mackey functor structure of the positive cone.

Definition 4.11. The *negative cone* is the subring of $\pi_\bigstar^G H\underline{\mathbb{F}_2}$ given by the direct sum of all $\pi_*^G(\Sigma^V H\underline{\mathbb{F}_2})$ for $V = p\sigma_1 + q\sigma_2 + r\sigma_3$ with $p, q, r \leq -1$.

If we are not in the positive cone or the negative cone, then we are in one of the six *mixed cones*, i.e. when $V = p\sigma_1 + q\sigma_2 + r\sigma_3$ is such that not all of p, q and r have the same sign.

Definition 4.12. If exactly one of p, q and r are negative then we are in one of the three mixed cones of *Type I*, and if exactly two of p, q and r are negative then we are in one of the three mixed cones of *Type II*.

We will only look at the positive cone in this section because it has the simplest ring structure, but our method of deriving this ring structure will give us insights into how we should be viewing the negative cone as well as the six mixed cones. Before we do this however, we introduce some further terminology (which applies not just to the positive cone).

Definition 4.13. We say that $\pi_i^G(\Sigma^V H\underline{\mathbb{F}_2})$ has *tridegree* (p,q,r) if $V = p\sigma_1 + q\sigma_2 + r\sigma_3$ and we say that it has *topological degree* i.

Moreover, we will refer to the triple complex (from Section 4.2) whose homology is $\pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$ as the triple complex at tridegree (p,q,r) and similarly for the topological degree.

For each $i \in \{1,2,3\}$, let x_i be the generator of $\pi_0^G(S^{\sigma_i} \wedge H\underline{\mathbb{F}_2}) \cong \mathbb{F}_2$ and let y_i be the generator of $\pi_1^G(S^{\sigma_i} \wedge H\underline{\mathbb{F}_2}) \cong \mathbb{F}_2$. By a similar argument to the ring structure in the C_2-equivariant case from Chapter 3, we know that the ring structure of $\pi_\bigstar^G H\underline{\mathbb{F}_2}$ when $\bigstar$ contains only non-negative multiples of σ_i (and does not contain non-zero multiples of σ_j for $j \neq i$) is given by the polynomial ring $\mathbb{F}_2[x_i, y_i]$. This can also be seen (assuming that $i = 1$) by using the ring structure of the positive cone in $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$ from Chapter 3 and that the restriction map $\mathrm{Res}_{H_2}^G$ is a ring map that is non-zero on non-zero elements in $\underline{\pi}_*(S^{p\sigma_1} \wedge H\underline{\mathbb{F}_2})$, where the restriction map being a ring map follows since $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is an $RO(G)$-graded Green functor.

One would hope that when we involve σ_1, σ_2 and σ_3 at once, then each non-zero class in homology is given by some product of these six classes, and indeed this is true modulo a single relation.

Theorem 4.14. *The positive cone in* $\pi_\bigstar^G H\underline{\mathbb{F}_2}$ *is given by the ring*

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3)},$$

with the classes x_i *and* y_i *for* $1 \leq i \leq 3$ *defined as above.*

Proof. Let C be the trigraded triple complex from the proof of Theorem 4.2 whose homology gives the positive cone. Then, we have a direct sum decomposition

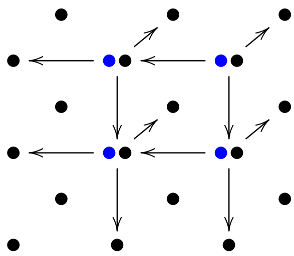
$$C = R_1 \oplus R_2,$$

where R_1 and R_2 are defined as follows. Using the notation from Section 4.2, the elements of R_1 are precisely the generators $z_{000} + z_{101} + z_{011} + z_{110}$ of each copy of $\mathbb{F}_2[2G/e]^G \cong \mathbb{F}_2^2$ in C. The elements of R_2 consist of the generators of each copy of $\mathbb{F}_2[G/G]^G$ and $\mathbb{F}_2[G/H_i]^G \cong \mathbb{F}_2$ for $i \in \{1, 2, 3\}$, the generators of each copy of $\mathbb{F}_2[G/e]^G \cong \mathbb{F}_2$, and the diagonal elements

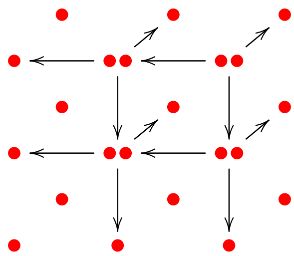
$$(z_{000} + z_{101} + z_{011} + z_{110}) + (z_{111} + z_{010} + z_{100} + z_{001})$$

of each copy of $\mathbb{F}_2[2G/e]^G \cong \mathbb{F}_2^2$ in C. This direct sum decomposition of our trigraded triple complex C is shown visually below.

R_1 :



R_2 :



Note that •• in the above diagram of the direct summand R_2 denotes only the diagonal element of the corresponding copy of $\mathbb{F}_2^2$.

As in the proof of Theorem 3.10, the ring structure of the positive cone is induced by taking the tensor product of the $\mathbb{F}_2[G]$-modules on the G/e levels of the trigraded complexes of Mackey functors and then taking fixed points. However, when we want to multiply elements of the subrings $\mathbb{F}_2[x_1, y_1]$, $\mathbb{F}_2[x_2, y_2]$ and $\mathbb{F}_2[x_3, y_3]$, we are only tensoring singly-graded complexes together. So, since for each $1 \leq i \leq 3$ we know that $x_i^{n_i} y_i^{m_i}$ is represented by the sum of the cosets $H_i + t_j H_i \in \mathbb{F}_2[G/H_i]^G$ (where $j \neq i$), it follows that for all $1 \leq i \neq j \leq 3$ we have that the product $x_i^{n_i} y_i^{m_i} x_j^{n_j} y_j^{m_j}$ is represented by the generator

$$(H_i, H_j) + (t_j H_i, H_j) + (H_i, t_i H_j) + (t_j H_i, t_i H_j) \in \mathbb{F}_2[G/H_1 \times G/H_2]^G,$$

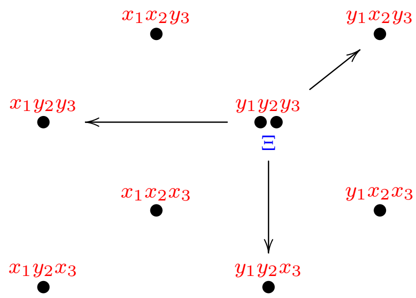
and the product $x_1^{n_1} y_1^{m_1} x_2^{n_2} y_2^{m_2} x_3^{n_3} y_3^{m_3}$ is represented by the diagonal element in $\mathbb{F}_2[2G/e]^G \cong \mathbb{F}_2^2$, where $n_i, m_i \geq 0$ for each $1 \leq i \leq 3$.

Therefore, if we let 1 denote the generator of the unique copy of $\mathbb{F}_2$ at tridegree $(0, 0, 0)$, it follows that

$$R_2 = \mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]\{1\}.$$

Now, let Ξ denote the unique element in R_1 at tridegree $(1, 1, 1)$. The portion of

C at tridegree $(1, 1, 1)$ is shown below.



So, letting $d = d^1 + d^2 + d^3$ as in the proof of Theorem 4.2, we see that

$$d(\Xi) = (x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3) \cdot 1.$$

Notice that we can view R_1 as an R_2-module generated by Ξ, i.e. that

$$R_1 = \mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]\{\Xi\}.$$

Indeed, we see that any element ξ in R_1 satisfies

$$d(\xi) = m(x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3) \cdot 1$$

for a unique monomial m in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$, and furthermore given any monomial m there is a unique such ξ, so by the Leibniz rule (noting that the trigraded triple complex C is a differential graded algebra under tensor product at the G/e level) we can therefore label ξ with $m \cdot \Xi$ since

$$\begin{aligned} d(m\Xi) &= d(m)\Xi + md(\Xi) \\ &= m(x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3) \cdot 1 \end{aligned}$$

as $d(m) = 0$. Hence, we have that the differential d is a map of $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$-modules determined by $d(\Xi) = (x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3) \cdot 1$. The positive cone is the homology of the chain complex

$$R_1 \xrightarrow{d} R_2$$

concentrated in degrees 0 and 1, so since this map is injective the homology is given by the cokernel of this map, which is precisely the ring

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3)}.$$

□

Using the result of Theorem 4.14 and that $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is an $RO(G)$-graded Green functor, we can now compute all the homotopy Mackey functors $\underline{\pi}_*(\Sigma^V H\underline{\mathbb{F}_2})$ for actual representations $V = p\sigma_1 + q\sigma_2 + r\sigma_3$. Note that we will give a more explicit computation of the homotopy Mackey functors in the positive cone in Section 4.7 that is generalisable in computing the homotopy Mackey functors in the negative and mixed cones, though in this section we show how the full Mackey functor structure of the positive cone can be derived with little computation.

Theorem 4.15. *The Mackey functor structure of the positive cone in* $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ *is given by the Mackey functor of* $RO(G)$*-graded rings*

$$\begin{array}{ccccc} & & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_1y_2y_3+y_1x_2y_3+y_1y_2x_3)} & & \\ & \swarrow & \downarrow & \searrow & \\ \frac{\mathbb{F}_2[y_1,x_2,y_2,x_3,y_3]}{(x_2y_3+y_2x_3)} & & \frac{\mathbb{F}_2[x_1,y_1,y_2,x_3,y_3]}{(x_1y_3+y_1x_3)} & & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(x_1y_2+y_1x_2)}, \\ & \searrow & \downarrow & \swarrow & \\ & & \mathbb{F}_2[y_1,y_2,y_3] & & \end{array}$$

where each restriction map is the identity on a generator of the domain that is also a generator of the codomain and is zero on a generator otherwise. The transfer maps are always zero.

Remark 4.16. Since the restriction maps in the above Mackey functor of $RO(G)$-graded rings are ring maps and the transfer maps are all zero, the Mackey functor is an $RO(G)$-graded Green functor.

Proof. We already know that the top level of this Mackey functor is the positive cone as an $RO(G)$-graded ring by Theorem 4.14. We first look at the restriction maps $\mathrm{Res}^G_{H_1}$, $\mathrm{Res}^G_{H_2}$ and $\mathrm{Res}^G_{H_3}$. By symmetry, it suffices to consider $\mathrm{Res}^G_{H_1}$ and we begin by looking at $\mathrm{Res}^G_{H_1}(x_i)$ and $\mathrm{Res}^G_{H_1}(y_i)$ for $i \in \{1,2,3\}$. The chain complex of Mackey functors computing $\underline{\pi}_*(S^{\sigma_1} \wedge H\underline{\mathbb{F}_2})$ is given by (noting that we only draw the G/G, G/H_1 and G/e levels)

$$\begin{array}{ccc} \overset{x_1}{\mathbb{F}_2} & \xleftarrow{\;0\;} & \overset{y_1}{\mathbb{F}_2} \\ 1 \downarrow \uparrow 0 & & \Delta \downarrow \uparrow \nabla \\ \mathbb{F}_2[G/G] & \xleftarrow{\;\nabla\;} & \mathbb{F}_2[G/H_1] \\ 1 \downarrow \uparrow 0 & & 1 \downarrow \uparrow 0 \\ \mathbb{F}_2[G/G] & \xleftarrow{\;\nabla\;} & \mathbb{F}_2[G/H_1]. \end{array}$$

So, we see that $\mathrm{Res}^G_{H_1}(x_1)$ is zero in homology but that $\mathrm{Res}^G_{H_1}(y_1)$ is non-zero in homology, and thus we will identify $\mathrm{Res}^G_{H_1}(y_1)$ with y_1. Similarly, the G/G, G/H_1 and G/e levels of the chain complex of Mackey functors computing $\underline{\pi}_*(S^{\sigma_2} \wedge H\underline{\mathbb{F}_2})$ is given by

$$\begin{array}{ccc}
\overset{x_2}{\mathbb{F}_2} & \xleftarrow{\quad 0 \quad} & \overset{y_2}{\mathbb{F}_2} \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} \\
\mathbb{F}_2 & \xleftarrow{\quad 0 \quad} & \mathbb{F}_2 \\
{\scriptstyle 1}\downarrow\uparrow{\scriptstyle 0} & & {\scriptstyle \Delta}\downarrow\uparrow{\scriptstyle \nabla} \\
\mathbb{F}_2[G/G] & \xleftarrow{\ \nabla\ } & \mathbb{F}_2[G/H_2].
\end{array}$$

Therefore, we see that $\mathrm{Res}^G_{H_1}(x_2)$ and $\mathrm{Res}^G_{H_1}(y_2)$ are both non-zero in homology, and we identify $\mathrm{Res}^G_{H_1}(x_2)$ with x_2 and $\mathrm{Res}^G_{H_1}(y_2)$ with y_2. Similarly, both $\mathrm{Res}^G_{H_1}(x_3)$ and $\mathrm{Res}^G_{H_1}(y_3)$ are non-zero in homology and we identify $\mathrm{Res}^G_{H_1}(x_3)$ with x_3 and $\mathrm{Res}^G_{H_1}(y_3)$ with y_3. Since $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is an $RO(G)$-graded Green functor, we know that $\mathrm{Res}^G_{H_1}$ is a ring map and thus for any monomial $x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}$ in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$ we have that

$$\mathrm{Res}^G_{H_1}(x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}) = \begin{cases} 0 & \text{if } i_1 > 0, \\ y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3} & \text{otherwise.} \end{cases}$$

Now, as in the statement of the theorem we want to show that the G/H_1 level of the positive cone in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is given by the $RO(G)$-graded ring

$$\frac{\mathbb{F}_2[y_1, x_2, y_2, x_3, y_3]}{(x_2y_3 + y_2x_3)}.$$

To do this, note that if $V = p\sigma_1 + q\sigma_2 + r\sigma_3$ is an actual representation, then as discussed in Section 4.2 we have for every k that

$$\pi_k^{H_1}(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2}) \cong \pi_k^{C_2}(\Sigma^{p+(q+r)\sigma}H\underline{\mathbb{F}_2}).$$

That is, the G/H_1 level of the homology at tridegree (p, q, r) is given by a shift (by p trivial suspensions) of the C_2-equivariant homology at degree $q + r$ from Chapter 3, i.e. unlike in Chapter 3 the middle levels of our homotopy Mackey functors carry some redundant gradings. In particular, recalling from Theorem 3.10 that the positive cone in $\pi_\bigstar^{C_2}H\underline{\mathbb{F}_2}$ is given by the polynomial ring $\mathbb{F}_2[x, y]$, the restriction $\mathrm{Res}^G_{H_1}(y_1)$ (which we also denote by y_1) is identified with the element $1 \in \mathbb{F}_2[x, y]$, the restrictions $\mathrm{Res}^G_{H_1}(x_2)$ and $\mathrm{Res}^G_{H_1}(x_3)$ (which we also denote by

x_2 and x_3 respectively) are both identified with x, and the restrictions $\mathrm{Res}^G_{H_1}(y_2)$ and $\mathrm{Res}^G_{H_1}(y_3)$ (which we also denote by y_2 and y_3 respectively) are both identified with y. Therefore, since x and y generate the positive cone in $\pi^{C_2}_\bigstar H\underline{\mathbb{F}_2}$ it follows that the restrictions y_1, x_2, y_2, x_3 and y_3 generate the G/H_1 level of the positive cone in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$.

Now, we know from Theorem 3.10 that the homology $\pi^{C_2}_*(\Sigma^{p+(q+r)\sigma}H\underline{\mathbb{F}_2})$ contains $q+r+1$ copies of $\mathbb{F}_2$ generated by $x^{q+r}, x^{q+r-1}y, \ldots, y^{q+r}$ in degrees $p, p+1, \ldots, p+q+r$ respectively. However, an arbitrary element in the homology $\pi^{H_1}_*(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})$ at the G/H_1 level is given by $y_1^p x_2^{q-k} y_2^k x_3^{r-\ell} y_3^\ell$, which as discussed above is identified with the element $x^{q+r-(k+\ell)}y^{k+\ell} \in \mathbb{F}_2[x,y]$. Assuming that at least one of k or r is strictly between 0 and $q+r$, we see that both $y_1^p x_2^{q-k} y_2^k x_3^{r-\ell} y_3^\ell$ and $y_1^p x_2^{q-(k-1)} y_2^{k-1} x_3^{r-(\ell+1)} y_3^{\ell+1}$ or both $y_1^p x_2^{q-k} y_2^k x_3^{r-\ell} y_3^\ell$ and $y_1^p x_2^{q-(k+1)} y_2^{k+1} x_3^{r-(\ell-1)} y_3^{\ell-1}$ are identified with $x^{q+r-(k+\ell)}y^{k+\ell} \in \mathbb{F}_2[x,y]$, which implies that

$$y_1^p x_2^{q-k} y_2^{k-1} x_3^{r-(\ell+1)} y_3^\ell (x_2y_3 + y_2x_3) = 0$$

and

$$y_1^p x_2^{q-(k+1)} y_2^k x_3^{r-\ell} y_3^{\ell-1} (x_2y_3 + y_2x_3) = 0$$

respectively. In particular, taking $p=0$ and $q=r=1$ we see that we have the relation $x_2y_3+y_2x_3=0$, and all other relations are multiples of this relation. Note that indeed the restriction map $\mathrm{Res}^G_{H_1}$ preserves the relation $x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3 = 0$ at the top level as

$$\mathrm{Res}^G_{H_1}(x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3) = y_1(x_2y_3 + y_2x_3) = 0.$$

Now that we have deduced the $RO(G)$-graded ring structure of the G/H_1 level of the positive cone in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ as well as the restriction map $\mathrm{Res}^G_{H_1}$, we next want to show that the transfer map $\mathrm{Tr}^G_{H_1}$ is zero in the positive cone. To do this, consider an arbitrary monomial $y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}$ in the ring $\mathbb{F}_2[y_1, x_2, y_2, x_3, y_3]/(x_2y_3+y_2x_3)$. Since the homotopy Mackey functors in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ are cohomological, we know that $\mathrm{Tr}^G_{H_1} \circ \mathrm{Res}^G_{H_1}$ is multiplication by the index $[G : H_1] = 2$, i.e. is the zero map. Hence, we see that

$$\begin{aligned}\mathrm{Tr}^G_{H_1}(y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}) &= \mathrm{Tr}^G_{H_1}(\mathrm{Res}^G_{H_1}(y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3})) \\ &= 0,\end{aligned}$$

which is precisely what we wanted to show. As mentioned before, all of the above reasoning is symmetric if we want to look instead at the subgroups H_2 or H_3.

Now, the fact that we can write the G/e level of the positive cone in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ as the $RO(G)$-graded ring $\mathbb{F}_2[y_1, y_2, y_3]$ follows since if we look at our above diagrams of the G/G, G/H_1 and G/e levels of the chain complexes of Mackey functors computing $\underline{\pi}_*(S^{\sigma_1} \wedge H\underline{\mathbb{F}_2})$ and $\underline{\pi}_*(S^{\sigma_2} \wedge H\underline{\mathbb{F}_2})$, we see that $\mathrm{Res}^{H_1}_e(x_i)$ for $i \in \{2, 3\}$ is zero in homology and $\mathrm{Res}^{H_1}_e(y_i)$ is non-zero in homology which we identify with y_i for each $i \in \{1, 2, 3\}$. The transfer and restriction maps between the G/H_i and G/e levels follow from the C_2-equivariant case given by Corollary 3.13. □

Remark 4.17. Note that in the proof of Theorem 4.15 there is an alternative argument for showing that the transfer maps are all zero that does not involve the concept of cohomological Mackey functors. Indeed, note that the element 1 in the $RO(G)$-graded ring $\frac{\mathbb{F}_2[y_1,x_2,y_2,x_3,y_3]}{(x_2y_3+y_2x_3)}$ is the restriction $\mathrm{Res}^G_{H_1}(1)$ of the element $1 \in \pi^G_0(S^0 \wedge H\underline{\mathbb{F}_2})$, and we know as discussed in Section 4.2 that $\underline{\pi}_0(S^0 \wedge H\underline{\mathbb{F}_2})$ is the constant Mackey functor $\underline{\mathbb{F}_2}$, which has zero transfers and thus $\mathrm{Tr}^G_{H_1}(1) = 0$. Since $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is an $RO(G)$-graded Green functor, we have by Frobenius reciprocity that

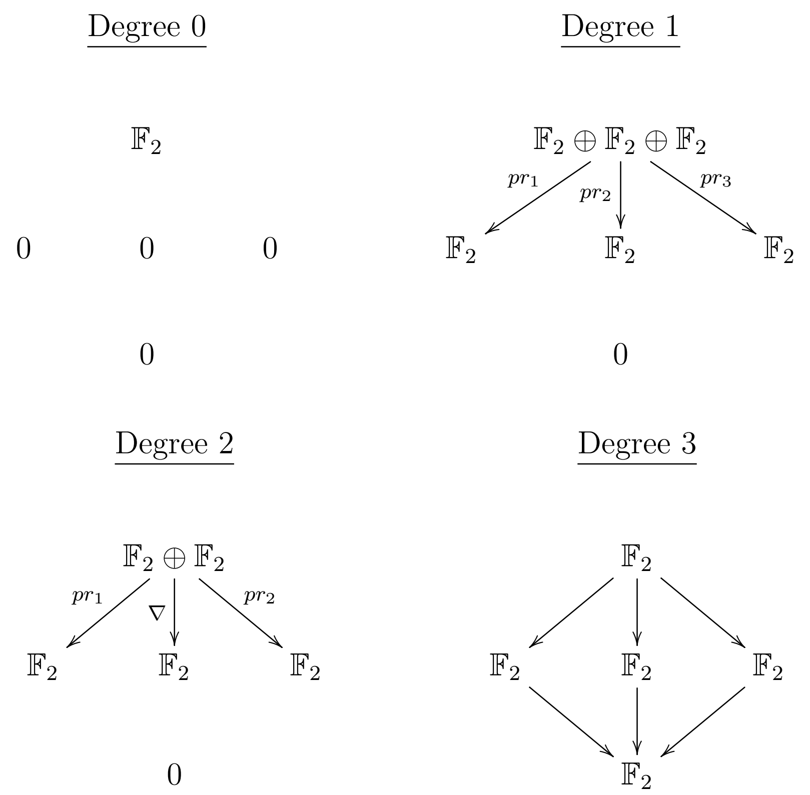
$$\begin{aligned}
\mathrm{Tr}^G_{H_1}(y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}) &= \mathrm{Tr}^G_{H_1}(1 \cdot y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}) \\
&= \mathrm{Tr}^G_{H_1}(1 \cdot \mathrm{Res}^G_{H_1}(y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3})) \\
&= \mathrm{Tr}^G_{H_1}(1) \cdot y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3} \\
&= 0.
\end{aligned}$$

We will now give an example to see how the Mackey functor of $RO(G)$-graded rings given by Theorem 4.15 can be used to explicitly write down homotopy Mackey functors in the positive cone.

Example 4.18. Suppose that we want to compute the homotopy Mackey functors $\underline{\pi}_*(\Sigma^{\sigma_1+\sigma_2+\sigma_3} H\underline{\mathbb{F}_2})$ at tridegree $(1, 1, 1)$. By Theorem 4.14 we know that the non-zero classes at the top level $\pi^G_*(\Sigma^{\sigma_1+\sigma_2+\sigma_3} H\underline{\mathbb{F}_2})$ are given as follows:

Degree 0	Degree 1	Degree 2	Degree 3
$x_1x_2x_3$	$y_1x_2x_3$	$\overline{y_1y_2x_3}$	$y_1y_2y_3$
	$x_1y_2x_3$	$\overline{x_1y_2y_3}$	
	$x_1x_2y_3$		

Here $\overline{x_1y_2y_3}$ and $\overline{y_1y_2x_3}$ are the classes (or cosets) in the positive cone represented by $x_1y_2y_3$ and $y_1y_2x_3$ respectively. By Theorem 4.15, we see that all restrictions of $x_1x_2x_3$ are zero. Furthermore, we see that only the restriction of $y_1x_2x_3$ to the subgroup H_1 is non-zero, and similarly only the restrictions of $x_1y_2x_3$ and $x_1x_2y_3$ to the subgroups H_2 and H_3 respectively are non-zero. Next, we see that only the restrictions of $\overline{y_1y_2x_3}$ to the subgroups H_1 and H_2 are non-zero, and only the restrictions of $\overline{x_1y_2y_3}$ to the subgroups H_2 and H_3 are non-zero. Finally, we see that all restrictions of $y_1y_2y_3$ are non-zero as there are no factors of x_1, x_2 or x_3. Therefore, we can deduce that the non-zero homotopy Mackey functors at tridegree $(1,1,1)$ are as follows:



Note that using the notation of Definition 2.20, the Mackey functor appearing in degree 1 is precisely the direct sum of the inflations of the constant Mackey functor given by

$$\phi^*_{H_1H_2H_3}\underline{\mathbb{F}_2} := \phi^*_{H_1}\underline{\mathbb{F}_2} \oplus \phi^*_{H_2}\underline{\mathbb{F}_2} \oplus \phi^*_{H_3}\underline{\mathbb{F}_2}.$$

In [3], the Mackey functors appearing in degrees 0, 1 and 2 are denoted by $\underline{g}$, $\phi^*_{LDR}\underline{\mathbb{F}_2}$ and $\underline{mg}$ respectively, noting that the authors denote the three C_2-subgroups of $G = C_2 \times C_2$ by L, D and R.

Finally, note that since we know all the homotopy Mackey functors in the positive cone, we also know all the homotopy Mackey functors in the negative cone by Anderson duality, as discussed in Section 4.3.

4.5 The negative and mixed cones

We now turn our attention to virtual representations. The multiplicative structure is more complicated in these cases, just as it was in the C_2-equivariant case in Chapter 3 when we discussed the negative cone in $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$. Again, we have three copies of the C_2-equivariant result given by $\pi_*^G(\Sigma^{p\sigma_1} H\underline{\mathbb{F}_2})$, $\pi_*^G(\Sigma^{q\sigma_2} H\underline{\mathbb{F}_2})$ and $\pi_*^G(\Sigma^{r\sigma_3} H\underline{\mathbb{F}_2})$. So, for each $i \in \{1,2,3\}$ let θ_i be the generator of $\pi_{-2}^G(S^{-2\sigma_i} \wedge H\underline{\mathbb{F}_2}) \cong \mathbb{F}_2$ which we know is infinitely divisible by x_i and y_i, i.e. is divisible by monomials in the graded polynomial ring $\mathbb{F}_2[x_i, y_i]$. As in Section 4.10, assuming that $i = 1$ this follows since the restriction map $\mathrm{Res}_{H_2}^G$ is non-zero on non-zero elements in $\pi_*^G(\Sigma^{-p\sigma_1} H\underline{\mathbb{F}_2})$ for $p \geq 2$ as seen in Section 4.2, so we can use that $\mathrm{Res}_{H_2}^G$ is a ring map as well as the ring structure of the negative cone in $\pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$, noting for example that $\mathrm{Res}_{H_2}^G(\theta_1)$ is identified with $\theta \in \pi_\bigstar^{C_2} H\underline{\mathbb{F}_2}$.

Unlike the ring structure of the positive cone given by Theorem 4.14, the negative and mixed cones contain classes that are not defined solely in terms of our previously defined classes from the three copies of the C_2-equivariant result in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$. Indeed, looking at the Poincaré series of Theorem 4.4 telling us the additive structure of the negative cone, we see (as in Example 4.7) that at tridegree $(-1,-1,-1)$ there is a single non-zero class with topological degree -3, and we will call this class

$$\Theta \in \pi_{-3}^G(S^{-\sigma_1-\sigma_2-\sigma_3} \wedge H\underline{\mathbb{F}_2}).$$

The reason why this is a 'new' class is that the homology at each tridegree $(-1,0,0)$, $(0,-1,0)$ and $(0,0,-1)$ is zero, i.e. we know in the C_2-equivariant case that $\pi_*^{C_2}(S^{-\sigma} \wedge H\underline{\mathbb{F}_2})$ is zero. Furthermore, looking at the additive structure of the mixed cones given by the Poincaré series in Theorems 4.5 and 4.6, we can immediately write down another six 'new' classes. Let

$$\kappa_1 \in \pi_1^G(S^{-\sigma_1+\sigma_2+\sigma_3} \wedge H\underline{\mathbb{F}_2})$$

$$\kappa_2 \in \pi_1^G(S^{\sigma_1-\sigma_2+\sigma_3} \wedge H\underline{\mathbb{F}_2})$$

$$\kappa_3 \in \pi_1^G(S^{\sigma_1+\sigma_2-\sigma_3} \wedge H\underline{\mathbb{F}_2})$$

be the unique non-zero classes (all of topological degree 1) at tridegrees $(-1,1,1)$, $(1,-1,1)$ and $(1,1,-1)$ respectively, and let

$$\iota_1 \in \pi_{-1}^G(S^{\sigma_1-\sigma_2-\sigma_3} \wedge H\underline{\mathbb{F}_2})$$

$$\iota_2 \in \pi_{-1}^G(S^{-\sigma_1+\sigma_2-\sigma_3} \wedge H\underline{\mathbb{F}_2})$$

$$\iota_3 \in \pi_{-1}^G(S^{-\sigma_1-\sigma_2+\sigma_3} \wedge H\underline{\mathbb{F}_2})$$

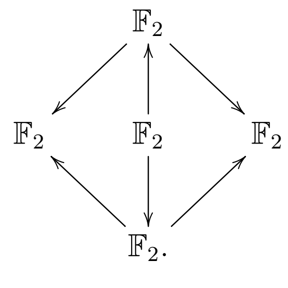
be the unique non-zero classes (all of topological degree -1) at tridegrees $(1,-1,-1)$, $(-1,1,-1)$ and $(-1,-1,1)$ respectively. The fact that each κ_i and ι_i for $i \in \{1,2,3\}$ is a 'new' class again follows since $\pi_*^{C_2}(S^{-\sigma} \wedge H\underline{\mathbb{F}_2})$ is zero. Now, we claim that we can in fact pass between these seven classes by multiplying by θ_1, θ_2 and θ_3. Note that these seven classes are important as the negative and mixed cones can be expressed entirely in terms of these seven classes as well as our previously defined classes from the three copies of the C_2-equivariant result in $\pi_\bigstar^G H\underline{\mathbb{F}_2}$.

Proposition 4.19. *For each* $\{i,j,k\} = \{1,2,3\}$*, we have that*

$$\iota_i\theta_i = \Theta \text{ and } \kappa_i\theta_j = \iota_k.$$

Proof. We first show that $\kappa_i\theta_j = \iota_k$ for each $\{i,j,k\} = \{1,2,3\}$. By symmetry, it suffices to show that $\kappa_1\theta_2 = \iota_3$, and by degree reasons we just need to show that $\kappa_1\theta_2$ is non-zero. Now, the homotopy Mackey functor $\underline{\kappa_1}$ corresponding to the class κ_1 is the constant Mackey functor $\underline{\mathbb{F}_2}$. Note that this Mackey functor can be derived by a similar argument to that of Example 4.7. That is, we look at the trigraded complex of Mackey functors at tridegree $(-1,1,1)$ and then take homology in turn in the σ_1, σ_2 and σ_3 directions and notice that the restriction map $\mathrm{Res}_{H_i}^G$ is non-zero for each $i \in \{1,2,3\}$, so each transfer map $\mathrm{Tr}_{H_i}^G$ must be zero as the composite $\mathrm{Res}_{H_i}^G \circ \mathrm{Tr}_{H_i}^G$ is zero. The non-zero restrictions between the G/H_i and G/e levels is as usual determined using the C_2-equivariant result given in Chapter 3.

Furthermore, as seen in Section 4.2 we have that the homotopy Mackey functor corresponding to the class θ_2 (namely $\underline{\pi}_{-2}(S^{-2\sigma_2} \wedge H\underline{\mathbb{F}_2})$) is given by

$\underline{\theta_2} =$ 

Now, in order to show that $\kappa_1\theta_2 \neq 0$ it suffices to show that

$$\mathrm{Res}_{H_3}^G(\kappa_1\theta_2) = \mathrm{Res}_{H_3}^G(\kappa_1)\mathrm{Res}_{H_3}^G(\theta_2) \neq 0,$$

where here we are using that the restriction map $\mathrm{Res}_{H_3}^G$ is a ring map. Looking at the two Mackey functors $\underline{\kappa_1}$ and $\underline{\theta_2}$, we see that $\mathrm{Res}_{H_3}^G(\kappa_1) \neq 0$ and $\mathrm{Res}_{H_3}^G(\theta_2) \neq 0$.

Thus, since

$$\pi_1^{H_3}(S^{-\sigma_1+\sigma_2+\sigma_3} \wedge H\underline{\mathbb{F}_2}) \cong \pi_1^{C_2}(S^1 \wedge H\underline{\mathbb{F}_2})$$

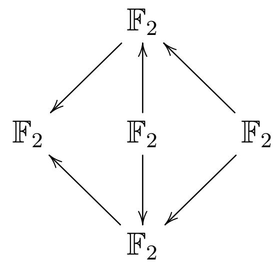
and

$$\pi_{-2}^{H_3}(S^{-2\sigma_2} \wedge H\underline{\mathbb{F}_2}) \cong \pi_{-2}^{C_2}(S^{-2\sigma} \wedge H\underline{\mathbb{F}_2}),$$

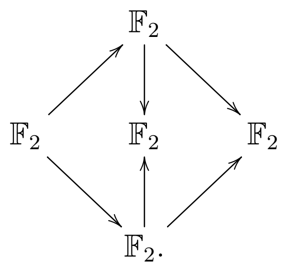
it follows that $\mathrm{Res}^G_{H_3}(\kappa_1)$ is identified with $1 \in \mathbb{F}_2[x, y]$ and $\mathrm{Res}^G_{H_3}(\theta_2)$ is identified with θ, and so indeed

$$\mathrm{Res}^G_{H_3}(\kappa_1)\mathrm{Res}^G_{H_3}(\theta_2) = 1 \cdot \theta = \theta \neq 0.$$

Next, we show that $\iota_i\theta_i = \Theta$ for all $i \in \{1, 2, 3\}$, and again by symmetry it suffices to show that $\iota_1\theta_1 = \Theta$. Recall from Example 4.7 that the homotopy Mackey functor corresponding to the class Θ is the dual constant Mackey functor $\underline{\mathbb{F}_2^*}$. The homotopy Mackey functors corresponding to ι_1 and θ_1 are given by

$\underline{\iota_1} =$ 

and

$\underline{\theta_1} =$ 

As before, the Mackey functor $\underline{\iota_1}$ is obtained by looking at the trigraded complex of Mackey functors at tridegree $(1, -1, -1)$ and then taking homology in turn in the σ_2, σ_3 and σ_1 directions and using that for each $i \in \{1, 2, 3\}$ the composite $\mathrm{Res}^G_{H_i} \circ \mathrm{Tr}^G_{H_i}$ is zero in this Mackey functor. Alternatively, the Mackey functor $\underline{\iota_1}$ can be obtained using our algebraic description in Section 4.7. We notice in particular that the transfer map $\mathrm{Tr}^G_{H_1}$ is non-zero in both the Mackey functors $\underline{\Theta}$ and $\underline{\theta_1}$ and that the restriction map $\mathrm{Res}^G_{H_1}$ is non-zero in the Mackey functor $\underline{\iota_1}$. Thus, since

$$\pi_{-3}^{H_1}(S^{-\sigma_1-\sigma_2-\sigma_3} \wedge H\underline{\mathbb{F}_2}) \cong \pi_{-2}^{C_2}(S^{-2\sigma} \wedge H\underline{\mathbb{F}_2}) = \theta,$$

$$\pi_{-1}^{H_1}(S^{\sigma_1-\sigma_2-\sigma_3} \wedge H\underline{\mathbb{F}_2}) \cong \pi_{-2}^{C_2}(S^{-2\sigma} \wedge H\underline{\mathbb{F}_2}) = \theta$$

and

$$\pi_{-2}^{H_1}(S^{-2\sigma_1} \wedge H\underline{\mathbb{F}_2}) \cong \pi_0^{C_2}(S^0 \wedge H\underline{\mathbb{F}_2}) = 1,$$

we have that

$$\pi_{-1}^{H_1}(S^{\sigma_1-\sigma_2-\sigma_3} \wedge H\underline{\mathbb{F}_2}) \cdot \pi_{-2}^{H_1}(S^{-2\sigma_1} \wedge H\underline{\mathbb{F}_2}) = \pi_{-3}^{H_1}(S^{-\sigma_1-\sigma_2-\sigma_3} \wedge H\underline{\mathbb{F}_2})$$

as $\theta \cdot 1 = \theta$. Hence, by Frobenius reciprocity it follows that

$$\begin{aligned}
\iota_1\theta_1 &= \iota_1 \cdot \mathrm{Tr}_{H_1}^G(\pi_{-2}^{H_1}(S^{-2\sigma_1} \wedge H\underline{\mathbb{F}_2})) \\
&= \mathrm{Tr}_{H_1}^G(\mathrm{Res}_{H_1}^G(\iota_1) \cdot \pi_{-2}^{H_1}(S^{-2\sigma_1} \wedge H\underline{\mathbb{F}_2})) \\
&= \mathrm{Tr}_{H_1}^G(\pi_{-1}^{H_1}(S^{\sigma_1-\sigma_2-\sigma_3} \wedge H\underline{\mathbb{F}_2}) \cdot \pi_{-2}^{H_1}(S^{-2\sigma_1} \wedge H\underline{\mathbb{F}_2})) \\
&= \mathrm{Tr}_{H_1}^G(\pi_{-3}^{H_1}(S^{-\sigma_1-\sigma_2-\sigma_3} \wedge H\underline{\mathbb{F}_2})) \\
&= \Theta.
\end{aligned}$$

□

Remark 4.20. By Proposition 4.19, we have in particular that $\kappa_i\theta_j\theta_k = \Theta$ for all $\{i, j, k\} = \{1, 2, 3\}$. Hence, since Θ is non-zero it follows that the products $\theta_j\theta_k$ for $j, k \in \{1, 2, 3\}$ with $j \neq k$ are non-zero. The result of Proposition 4.19 can also allow us to think of the class Θ as being divisible by θ_1, θ_2 and θ_3, where

$$\iota_i = \frac{\Theta}{\theta_i} \text{ and } \kappa_i = \frac{\Theta}{\theta_j\theta_k}$$

for each $\{i, j, k\} = \{1, 2, 3\}$.

We also have the following corollaries of Proposition 4.19 telling us when various products involving the seven classes Θ, ι_1, ι_2, ι_3, κ_1, κ_2 and κ_3 are zero.

Corollary 4.21. *If $i, j \in \{1, 2, 3\}$ are such that $i \neq j$, then*

$$\iota_i\theta_j = 0.$$

Proof. Let $k \in \{1, 2, 3\}$ be such that $\{i, j, k\} = \{1, 2, 3\}$. Then, by Proposition 4.19 we know that $\iota_i = \kappa_k\theta_j$. However, this implies that

$$\iota_i\theta_j = \kappa_k\theta_j^2 = 0$$

as we know that $\theta_j^2 = 0$. □

Corollary 4.22. *For each* $i \in \{1, 2, 3\}$*, we have that*

$$\iota_i \kappa_i = 0.$$

Proof. Suppose for the sake of a contradiction that $\iota_i \kappa_i$ is non-zero. Then, by degree reasons we have that $\iota_i \kappa_i = 1$ where 1 denotes the single non-zero class at tridegree $(0, 0, 0)$ (which has topological degree 0). Let $j \in \{1, 2, 3\}$ be such that $j \neq i$. Then, multiplying both sides of the equation $\iota_i \kappa_i = 1$ by θ_j gives $\iota_i \kappa_i \theta_j = \theta_j$. However,

$$\iota_i \kappa_i \theta_j = \kappa_i (\iota_i \theta_j) = 0$$

as $\iota_i \theta_j = 0$ by Corollary 4.21. Hence, since θ_j on the right-hand side is a non-zero class, we have indeed reached a contradiction. □

Corollary 4.23. *For each* $i \in \{1, 2, 3\}$*, we have that*

$$\Theta^2 = \theta_i \Theta = \kappa_i \Theta = \iota_i \Theta = 0.$$

Proof. Let $i \in \{1, 2, 3\}$ be arbitrary. By Proposition 4.19, we can write $\Theta = \iota_i \theta_i$. So, since $\theta_i^2 = 0$ we have that

$$\Theta^2 = \iota_i^2 \theta_i^2 = 0,$$

and similarly

$$\theta_i \Theta = \theta_i \iota_i \theta_i = \iota_i \theta_i^2 = 0.$$

Furthermore, since $\kappa_i \iota_i = 0$ by Corollary 4.22, we have that

$$\kappa_i \Theta = \kappa_i \iota_i \theta_i = 0.$$

Finally, to show that $\iota_i \Theta = 0$ choose $j, k \in \{1, 2, 3\}$ so that $\{i, j, k\} = \{1, 2, 3\}$. By Proposition 4.19, we have that $\iota_i = \kappa_j \theta_k$, and so

$$\iota_i \Theta = \kappa_j \theta_k \Theta = \theta_k (\kappa_j \Theta) = 0$$

as we know from above that $\kappa_j \Theta = 0$. □

Corollary 4.24. *For every* $i, j \in \{1, 2, 3\}$*, we have that*

$$\iota_i \iota_j = 0.$$

Proof. We first show that $\iota_i^2 = 0$. Choose $j, k \in \{1, 2, 3\}$ so that $\{i, j, k\} = \{1, 2, 3\}$. By Proposition 4.19, we have that $\iota_i = \kappa_j \theta_k$. Then, it follows that

$$\iota_i^2 = \kappa_j^2 \theta_k^2 = 0$$

as $\theta_k^2 = 0$. Next, suppose that $i, j \in \{1, 2, 3\}$ are such that $i \neq j$. Choose $k \in \{1, 2, 3\}$ so that $\{i, j, k\} = \{1, 2, 3\}$. By Proposition 4.19, we have that $\iota_i = \kappa_j \theta_k$ and $\iota_j = \kappa_i \theta_k$. Therefore, it follows that

$$\iota_i \iota_j = \kappa_j \theta_k \kappa_i \theta_k = \kappa_j \kappa_i \theta_k^2 = 0$$

as $\theta_k^2 = 0$. □

Furthermore, by the result of Proposition 4.19 we can now understand the non-zero homology classes appearing in tridegrees $(-p, -q, -1)$, $(-p, -1, -r)$ and $(-1, -q, -r)$ for $p, q, r \geq 1$. Again since $\pi_*^{C_2}(S^{-\sigma} \wedge H\underline{\mathbb{F}_2}) = 0$, we know that the homology classes in these tridegrees are not products of the classes coming from the three copies of the C_2-equivariant calculation in $\pi_\bigstar^G H\underline{\mathbb{F}_2}$.

Proposition 4.25. *The non-zero homology classes in tridegrees* $(-1, -q, -r)$ *for* $q, r \geq 1$ *are given by the products*

$$\kappa_1 \cdot \frac{\theta_2}{x_2^{i_2} y_2^{j_2}} \cdot \frac{\theta_3}{x_3^{i_3} y_3^{j_3}}$$

for $x_2^{i_2} y_2^{j_2}$ *a monomial in* $\mathbb{F}_2[x_2, y_2]$ *and* $x_3^{i_3} y_3^{j_3}$ *a monomial in* $\mathbb{F}_2[x_3, y_3]$.

Proof. Fix some arbitrary $q, r \geq 1$ and let $x_2^{i_2} y_2^{j_2}$ and $x_3^{i_3} y_3^{j_3}$ be monomials in $\mathbb{F}_2[x_2, y_2]$ and $\mathbb{F}_2[x_3, y_3]$ respectively with $i_2 + j_2 = -q$ and $i_3 + j_3 = -r$. By Proposition 4.19, we know that $\kappa_1 \theta_2 \theta_3 = \Theta$. However, we also know that there are non-zero classes $\frac{\theta_2}{x_2^{i_2} y_2^{j_2}}$ and $\frac{\theta_3}{x_3^{i_3} y_3^{j_3}}$ in homology such that

$$\frac{\theta_2}{x_2^{i_2} y_2^{j_2}} \cdot x_2^{i_2} y_2^{j_2} = \theta_2 \text{ and } \frac{\theta_3}{x_3^{i_3} y_3^{j_3}} \cdot x_3^{i_3} y_3^{j_3} = \theta_3.$$

Hence, it follows that $\kappa_1 \frac{\theta_2}{x_2^{i_2} y_2^{j_2}} \frac{\theta_3}{x_3^{i_3} y_3^{j_3}}$ is a non-zero class in homology at tridegree $(-1, -q, -r)$ such that we get back the class Θ after multiplying by the monomial $x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}$. Furthermore, notice that if $x_2^{i_2'} y_2^{j_2'}$ and $x_3^{i_3'} y_3^{j_3'}$ are monomials in $\mathbb{F}_2[x_2, y_2]$ and $\mathbb{F}_2[x_3, y_3]$ with $i_2' + j_2' = -q$ and $i_3' + j_3' = -r$ such that $x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3} \neq x_2^{i_2'} y_2^{j_2'} x_3^{i_3'} y_3^{j_3'}$, then

$$\kappa_1 \frac{\theta_2}{x_2^{i_2} y_2^{j_2}} \frac{\theta_3}{x_3^{i_3} y_3^{j_3}} \neq \kappa_1 \frac{\theta_2}{x_2^{i_2'} y_2^{j_2'}} \frac{\theta_3}{x_3^{i_3'} y_3^{j_3'}}.$$

Indeed, assume without loss of generality that $i_2 > i_2'$. Then, we see that

$$\kappa_1 \frac{\theta_2}{x_2^{i_2} y_2^{j_2}} \frac{\theta_3}{x_3^{i_3} y_3^{j_3}} \cdot x_2^{i_2} = \kappa_1 \frac{\theta_2}{y_2^{j_2}} \frac{\theta_3}{x_3^{i_3} y_3^{j_3}}$$

is non-zero, but

$$\kappa_1 \frac{\theta_2}{x_2^{i_2'} y_2^{j_2'}} \frac{\theta_3}{x_3^{i_3'} y_3^{j_3'}} \cdot x_2^{i_2} = 0.$$

By Theorem 4.4, we know that the Poincaré series for $\pi_*^G(S^{-\sigma_1 - q\sigma_2 - r\sigma_3} \wedge H\underline{\mathbb{F}_2})$ is given by

$$\frac{1}{x^{q+r+1}}(1 + x + \cdots + x^{q-1})(1 + x + \cdots + x^{r-1}),$$

so the above products form the full tridegree $(-1, -q, -r)$. □

Remark 4.26. By the proof of Proposition 4.25, we can therefore think of the class Θ as being infinitely divisible by x_i and y_i for each $i \in \{1, 2, 3\}$ and we can write

$$\kappa_1 \frac{\theta_2}{x_2^{i_2} y_2^{j_2}} \frac{\theta_3}{x_3^{i_3} y_3^{j_3}} = \frac{\Theta}{x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}},$$

which we will do in Section 4.6.

Although as mentioned earlier it is clear that no non-zero class in tridegrees $(-p, -q, -1)$, $(-p, -1, -r)$ and $(-1, -q, -r)$ for $p, q, r \geq 1$ is a product of classes from the three copies of the C_2-equivariant result in $\pi_\bigstar^G H\underline{\mathbb{F}_2}$, the same is in fact true in the full negative cone.

Proposition 4.27. *If $x_1^{i_1} y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}$ is a monomial in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$, then*

$$\frac{\theta_1}{x_1^{i_1} y_1^{j_1}} \cdot \frac{\theta_2}{x_2^{i_2} y_2^{j_2}} \cdot \frac{\theta_3}{x_3^{i_3} y_3^{j_3}} = 0.$$

Proof. We proceed by induction on j_1. If $j_1 = 0$, then we notice that

$$\frac{\theta_1}{x_1^{i_1}} \frac{\theta_2}{x_2^{i_2} y_2^{j_2}} \frac{\theta_3}{x_3^{i_3} y_3^{j_3}} = (x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3) \frac{\theta_1}{x_1^{i_1+1}} \frac{\theta_2}{x_2^{i_2} y_2^{j_2+1}} \frac{\theta_3}{x_3^{i_3} y_3^{j_3+1}}$$

as $y_1 \frac{\theta_1}{x_1^{i_1+1}} = 0$. However, the right-hand side is zero as we know from Theorem 4.14 that

$$x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3 = 0.$$

Similarly, given an arbitrary $j_1 > 0$, we see that

$$\begin{aligned}&(x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3)\frac{\theta_1}{x_1^{i_1+1}y_1^{j_1}}\frac{\theta_2}{x_2^{i_2}y_2^{j_2+1}}\frac{\theta_3}{x_3^{i_3}y_3^{j_3+1}}\\ &= \frac{\theta_1}{x_1^{i_1}y_1^{j_1}}\frac{\theta_2}{x_2^{i_2}y_2^{j_2}}\frac{\theta_3}{x_3^{i_3}y_3^{j_3}} + \frac{\theta_1}{x_1^{i_1+1}y_1^{j_1-1}}\frac{\theta_2}{x_2^{i_2-1}y_2^{j_2+1}}\frac{\theta_3}{x_3^{i_3}y_3^{j_3}} + \frac{\theta_1}{x_1^{i_1+1}y_1^{j_1-1}}\frac{\theta_2}{x_2^{i_2}y_2^{j_2}}\frac{\theta_3}{x_3^{i_3-1}y_3^{j_3+1}}\\ &= \frac{\theta_1}{x_1^{i_1}y_1^{j_1}}\frac{\theta_2}{x_2^{i_2}y_2^{j_2}}\frac{\theta_3}{x_3^{i_3}y_3^{j_3}}\end{aligned}$$

where the last equality follows by the induction hypothesis, so this product is zero. □

By degree reasons we know that $\Theta x_i = \Theta y_i = 0$ for all $i \in \{1, 2, 3\}$ and similarly $\iota_i x_j = \iota_i y_j = 0$ for all $i, j \in \{1, 2, 3\}$ with $i \neq j$. However, it is not true that $\kappa_i x_i = \kappa_i y_i = 0$ for all $i \in \{1, 2, 3\}$.

Proposition 4.28. *For each $\{i, j, k\} = \{1, 2, 3\}$, we have that*

$$\kappa_i x_i = x_j y_k + y_j x_k \text{ and } \kappa_i y_i = y_j y_k.$$

Proof. By symmetry, it suffices to compute the products $\kappa_1 x_1$ and $\kappa_1 y_1$. To do this, recall from the proof of Proposition 4.19 that the homotopy Mackey functor corresponding to κ_1 is the constant Mackey functor $\underline{\mathbb{F}_2}$. Similarly, we know from Theorem 4.15 that the homotopy Mackey functor corresponding to y_1 is also given by $\underline{\mathbb{F}_2}$. So, since κ_1 lives in tridegree $(-1, 1, 1)$ with topological degree 1 and y_1 lives in tridegree $(1, 0, 0)$ with topological degree 1, we can identify $\mathrm{Res}^G_{H_2}(\kappa_1)$ with $1 \in \mathbb{F}_2[x, y]$ and $\mathrm{Res}^G_{H_2}(y_1)$ with $y \in \mathbb{F}_2[x, y]$, and so

$$\mathrm{Res}^G_{H_2}(\kappa_1 y_1) = \mathrm{Res}^G_{H_2}(\kappa_1)\mathrm{Res}^G_{H_2}(y_1) = y \neq 0,$$

which implies that $\kappa_1 y_1 \neq 0$. Since the product $\kappa_1 y_1$ lives in tridegree $(0, 1, 1)$ with topological degree 2, it follows from the ring structure of the positive cone given by Theorem 4.14 that

$$\kappa_1 y_1 = y_2 y_3.$$

Similarly, we know from Theorem 4.15 that $\mathrm{Res}^G_{H_2}(x_1)$ is non-zero, so we can identify $\mathrm{Res}^G_{H_2}(x_1)$ with $x \in \mathbb{F}_2[x, y]$, and thus

$$\mathrm{Res}^G_{H_2}(\kappa_1 x_1) = \mathrm{Res}^G_{H_2}(\kappa_1)\mathrm{Res}^G_{H_2}(x_1) = x \neq 0,$$

which implies that $\kappa_1 x_1 \neq 0$. However, the product $\kappa_1 x_1$ lives in tridegree $(0, 1, 1)$ with topological degree 1, so in this case we just know that it is some non-zero element in the copy of $\mathbb{F}_2^2$ generated by x_2y_3 and y_2x_3. But since $\kappa_1 y_1 = y_2y_3$ we have in particular that

$$\kappa_1 x_1 y_1 = x_1 y_2 y_3,$$

and hence using the relation $x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3 = 0$ it follows that

$$\kappa_1 x_1 y_1 = y_1(x_2y_3 + y_2x_3).$$

Therefore, we must have that

$$\kappa_1 x_1 = x_2y_3 + y_2x_3$$

as otherwise we would have that $\kappa_1x_1y_1 = y_1x_2y_3$ or $\kappa_1x_1y_1 = y_1y_2x_3$, but neither of these are equal to $y_1(x_2y_3 + y_2x_3)$ by Theorem 4.14. □

By similar arguments to the proof of Proposition 4.28 using that the restriction maps are ring maps and that the homotopy Mackey functors corresponding to each κ_i is the constant Mackey functor $\underline{\mathbb{F}_2}$, we have that κ_i^2 is non-zero and $\kappa_i\kappa_j = y_k^2$ for each $\{i, j, k\} = \{1, 2, 3\}$. Note that we cannot express κ_1^2, κ_2^2 and κ_3^2 in terms of the x_i, y_i and θ_i classes. Indeed, if this were the case then since κ_1^2 lives in tridegree $(-2, 2, 2)$ it would have to contain a factor of θ_1, but $\theta_1x_1 = 0$ whereas $\kappa_1^2x_1 = \kappa_1x_2y_3 + \kappa_1y_2x_3$ is non-zero. An alternative expression for κ_1^2 is given in Section 4.6.

Corollary 4.29. *Given any monomial m in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$, there is a non-zero class η_m in the negative cone such that $\eta_m \cdot m = \Theta$.*

Proof. Fix a monomial $x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}$ in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$, and consider the product

$$\kappa_1^{i_1+j_1+1}\frac{\theta_2}{x_2^{i_1+i_2}y_2^{j_1+j_2}}\frac{\theta_3}{x_3^{i_3}y_3^{i_1+j_1+j_3}}.$$

If we first multiply this product by $x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}$, then we are left with

$$\kappa_1^{i_1+j_1+1}\frac{\theta_2}{x_2^{i_1}y_2^{j_1}}\frac{\theta_3}{y_3^{i_1+j_1}}.$$

By inductively using that $\kappa_1x_1 = x_2y_3 + y_2x_3$ and $\kappa_1y_1 = y_2y_3$ from Proposition 4.28, if we multiply the above by $x_1^{i_1}y_1^{j_1}$ we get $\kappa_1\theta_2\theta_3$ which is equal to Θ by Proposition 4.19. □

Note that if η is a class in the negative cone such that $\eta \cdot x_1y_2y_3m = \Theta$ for some monomial m in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$, then by the relation $x_1y_2y_3+y_1x_2y_3+y_1y_2x_3 = 0$ in the positive cone it follows that

$$\eta \cdot y_1x_2y_3m + \eta \cdot y_1y_2x_3m = \Theta$$

and hence we must also have that either $\eta \cdot y_1x_2y_3m = \Theta$ or $\eta \cdot y_1y_2x_3m = \Theta$, but not both. This discussion is of course symmetric if we instead first assume that $\eta \cdot y_1x_2y_3m = \Theta$ or $\eta \cdot y_1y_2x_3m = \Theta$. For example, if we multiply the product

$$\kappa_1^2 \frac{\theta_2}{x_2y_2} \frac{\theta_3}{y_3^2}$$

by either $x_1y_2y_3$ or $y_1x_2y_3$ then we get back Θ, though we get zero when we multiply by $y_1y_2x_3$. The homology in the negative cone (and indeed in the mixed cones) will be discussed more explicitly in Section 4.6.

4.6 An algebraic description of the homology

In this section we will give a complete algebraic description of the homology $\pi_\bigstar^G H\underline{\mathbb{F}_2}$. In particular, we explain how the homology in the positive, negative and mixed cones can be expressed entirely in terms of the classes x_i, y_i, θ_i, κ_i, ι_i and Θ for $i \in \{1, 2, 3\}$, involving the polynomial

$$f = x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3$$

seen earlier in this chapter. Note that by Proposition 4.19, we can then express the homology entirely in terms of the x_i, y_i, θ_i and κ_i classes for $i \in \{1, 2, 3\}$, but for ease of notation (and the fact that for example $\iota_1 = \kappa_2\theta_3 = \kappa_3\theta_2$), we will continue using the notation ι_i and Θ. First, recall from the proof of Theorem 4.14 that we have a direct sum decomposition of the trigraded triple complex computing the homology in the positive cone given by

$$\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]\{\Xi\} \oplus \mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]\{1\},$$

where Ξ lives in tridegree $(1, 1, 1)$ and topological degree 3, and the differential d takes the first summand to the second summand. In particular, we saw that

$$d(\Xi) = (x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3) \cdot 1,$$

and the homology in the positive cone is given by the homology of the chain complex

$$\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]\{\Xi\} \xrightarrow{f} \mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]\{1\}$$

concentrated in two degrees, where f is the polynomial as above. This map is injective but not surjective, so the homology in the positive cone is given by the cokernel of this map, namely the quotient ring

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3)}.$$

Now, we claim that similar analysis can be done for the negative cone and the six mixed cones, and we begin by looking at the negative cone. As in Section 4.5, let Θ denote the unique non-zero homology class at tridegree $(-1, -1, -1)$ and topological degree -3. We will be considering the $\mathbb{F}_2$-module

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1^\infty, y_1^\infty, x_2^\infty, y_2^\infty, x_3^\infty, y_3^\infty)}\{\Theta\}.$$

The notation here is similar to that used in Chapter 3, and this set is the $\mathbb{F}_2$-linear span of elements of the form

$$\frac{\Theta}{x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}}$$

where $x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}$ is a monomial in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$. Note that each of these elements do not necessarily represent homology classes, rather they will be used to label elements in the trigraded triple complex computing the homology in the negative cone. However, various sums of these elements will be homology classes as discussed in Remark 4.26. We will also consider the $\mathbb{F}_2$-module

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1^\infty, y_1^\infty, x_2^\infty, y_2^\infty, x_3^\infty, y_3^\infty)}\{\theta_1\theta_2\theta_3\}$$

where the homology classes θ_1, θ_2 and θ_3 are defined as in Section 4.5, and this set is the $\mathbb{F}_2$-linear span of elements of the form

$$\frac{\theta_1\theta_2\theta_3}{x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}}$$

which are indeed all homology classes, namely the product of the homology classes

$$\frac{\theta_1}{x_1^{i_1}y_1^{j_1}} \cdot \frac{\theta_2}{x_2^{i_2}y_2^{j_2}} \cdot \frac{\theta_3}{x_3^{i_3}y_3^{j_3}}.$$

In order to make sense of the following theorem, we will use how products of this form behave in homology when multiplied by monomials in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$, for example that we get zero when the monomial does not divide $x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}$.

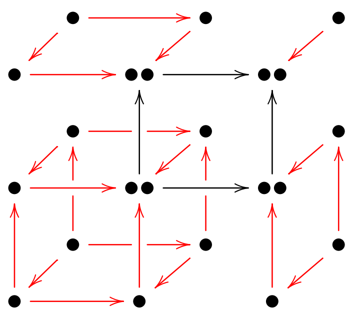
Theorem 4.30. *The homology in the negative cone is given by the homology of the chain complex*

$$\frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty,x_3^\infty,y_3^\infty)}\{\Theta\} \xrightarrow{f} \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty,x_3^\infty,y_3^\infty)}\{\theta_1\theta_2\theta_3\}$$

concentrated in two degrees, given by multiplication by the polynomial $f = x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3$. More precisely, this map is given by

$$\frac{\Theta}{x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}} \mapsto f\cdot\frac{\theta_1\theta_2\theta_3}{x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}}.$$

Proof. We first consider a direct sum decomposition $D\oplus X$ of the trigraded triple complex computing the homology in the negative cone defined as follows. The elements forming the direct summand D are the domain and image of any of the three differentials d^1, d^2 and d^3 whose domain is a single copy of $\mathbb{F}_2$. In the following diagram, the direct summand D is the domain and image of all the red differentials.



Letting $d = d^1 + d^2 + d^3$ as usual be the total differential in our trigraded triple complex, we have that the homology in the negative cone is given by the homology of the chain complex

$$D\oplus X \xrightarrow{d} D\oplus X.$$

Since the differential d maps D onto itself and maps X to both D and X, we can write the differential d as the matrix

$$\begin{bmatrix} d_D & d_D^X \\ 0 & d_X \end{bmatrix}$$

with respect to the direct sum decomposition $D\oplus X$. Now, notice that we have the short exact sequence of chain complexes (each concentrated in two degrees)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D & \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} & D \oplus X & \xrightarrow{[0\ \ 1]} & X & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle d_D} & & \downarrow{\scriptstyle \begin{bmatrix} d_D & d_D^X \\ 0 & d_X \end{bmatrix}} & & \downarrow{\scriptstyle d_X} & & \downarrow \\
0 & \longrightarrow & D & \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} & D \oplus X & \xrightarrow{[0\ \ 1]} & X & \longrightarrow & 0,
\end{array}$$

so we get a long exact sequence in homology. However, by iteratively using the spectral sequence of a double complex at each tridegree, we see that the homology of the chain complex $D \xrightarrow{d_D} D$ is zero. Therefore, it follows that the homology in the negative cone is isomorphic to the homology of the chain complex

$$X \xrightarrow{d_X} X.$$

That is, in the triple complex at each tridegree in the negative cone we can ignore all elements contained in the direct summand D, so we only consider the trigraded triple complex X. Now, we break X into a direct sum

$$X = S_1 \oplus S_2$$

as we did in the proof of Theorem 4.14. At each fixed tridegree in X, the (non-zero) $\mathbb{F}_2$-modules in each position of the corresponding triple complex are copies of either $\mathbb{F}_2$ or $\mathbb{F}_2^2$. The copies of $\mathbb{F}_2$ at each tridegree in X as well as the copies of $\mathbb{F}_2$ contained in each $\mathbb{F}_2^2$ generated by the elements $(1,0) \in \mathbb{F}_2^2$ corresponding to the sum

$$z_{000} + z_{101} + z_{011} + z_{110}$$

form the direct summand S_1. The copies of $\mathbb{F}_2$ contained in each $\mathbb{F}_2^2$ generated by the diagonal elements $(1,1) \in \mathbb{F}_2^2$ corresponding to the sum

$$(z_{000} + z_{101} + z_{011} + z_{110}) + (z_{111} + z_{010} + z_{100} + z_{001})$$

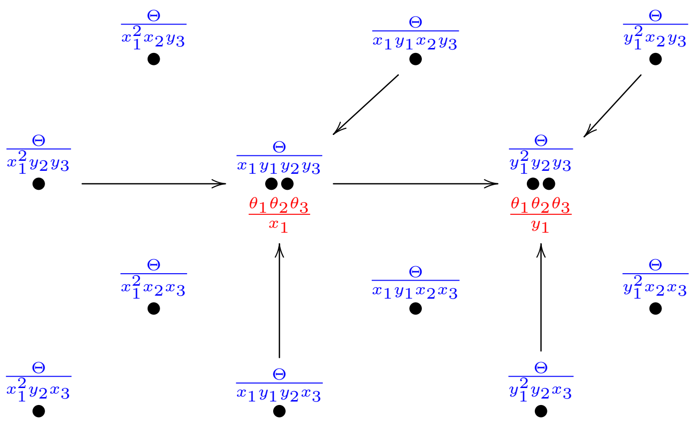
form the direct summand S_2. Now, by degree reasons each element in S_1 can be labelled with a unique element in

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1^\infty, y_1^\infty, x_2^\infty, y_2^\infty, x_3^\infty, y_3^\infty)}\{\Theta\}$$

and every element in S_2 is a unique homology class in

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1^\infty, y_1^\infty, x_2^\infty, y_2^\infty, x_3^\infty, y_3^\infty)}\{\theta_1\theta_2\theta_3\}.$$

For example, the following diagram shows the trigraded triple complex $X = S_1 \oplus S_2$ at tridegree $(-3,-2,-2)$.



As in the proof of Theorem 4.14 we have that the total differential d in the trigraded triple complex X takes the summand S_1 to the summand S_2. In particular, we see that the total differential is given by multiplication by $f = x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3$ with respect to our labelling of elements in S_1 and S_2. Indeed, given $i_1, j_1, i_2, j_2, i_3, j_3 \geq 0$ we see that

$$d\left(\frac{\Theta}{x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}}\right)$$
$$= \frac{\theta_1\theta_2\theta_3}{x_1^{i_1-1}y_1^{j_1}x_2^{i_2}y_2^{j_2-1}x_3^{i_3}y_3^{j_3-1}} + \frac{\theta_1\theta_2\theta_3}{x_1^{i_1}y_1^{j_1-1}x_2^{i_2-1}y_2^{j_2}x_3^{i_3}y_3^{j_3-1}} + \frac{\theta_1\theta_2\theta_3}{x_1^{i_1}y_1^{j_1-1}x_2^{i_2}y_2^{j_2-1}x_3^{i_3-1}y_3^{j_3}},$$

where we interpret any of these terms with a negative power of x_i or y_i in the denominator as zero. Using how homology classes in S_2 behave under multiplication by elements in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$ we see that the above is equal to

$$f \cdot \frac{\theta_1\theta_2\theta_3}{x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}},$$

so we are done since the homology in the negative cone is given by the homology of the chain complex

$$S_1 \xrightarrow{d} S_2.$$

□

Notice in particular that the map in the statement of Theorem 4.30 is surjective, which can be seen for example by Proposition 4.27. Furthermore, as discussed in Section 4.5 we can indeed view elements in the kernel of this map as classes in the negative cone such that we get back the class Θ after multiplying by a monomial in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$. For example, we can think of the element

$$\frac{\Theta}{x_1y_2y_3} + \frac{\Theta}{y_1x_2y_3}$$

in the kernel as a homology class such that we get back the class Θ after multiplying by $x_1y_2y_3$ or $y_1x_2y_3$, and hence zero after multiplying by $y_1y_2x_3$ as $x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3 = 0$ in homology. That is, we can identify it with the product

$$\kappa_1^2 \frac{\theta_2}{x_2y_2} \frac{\theta_3}{y_3^2}$$

using Proposition 4.28. We now look at the six mixed cones, and the arguments in these cases will be similar to Theorem 4.30. We first consider the mixed cones of Type I. For each $i \in \{1, 2, 3\}$, consider the homology class κ_i defined in Section 4.5. If $i = 1$, we will be looking now at the $\mathbb{F}_2$-module

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1^\infty, y_1^\infty)}\{\kappa_1\}$$

spanned by elements of the form

$$\frac{\kappa_1}{x_1^{i_1} y_1^{j_1}} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}$$

with $x_1^{i_1} y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}$ a monomial in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$. As before, not all elements in this $\mathbb{F}_2$-module will represent homology classes, but every element in the $\mathbb{F}_2$-module

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1^\infty, y_1^\infty)}\{\theta_1\}$$

is indeed a homology class, however not every element will represent a non-zero homology class.

Theorem 4.31. *The homology in the mixed cone of Type I corresponding to tridegrees $(-p, q, r)$ with $p \geq 1$ and $q, r \geq 0$ is given by the homology of the chain complex*

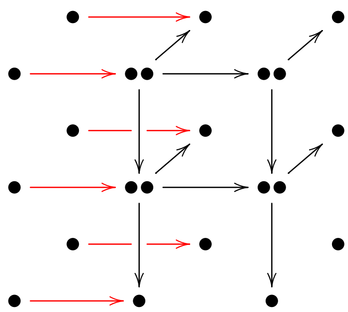
$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1^\infty, y_1^\infty)}\{\kappa_1\} \xrightarrow{f} \frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1^\infty, y_1^\infty)}\{\theta_1\}$$

concentrated in two degrees, given by multiplication by the polynomial $f = x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3$. More precisely, the map is given by

$$\frac{\kappa_1}{x_1^{i_1} y_1^{j_1}} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3} \mapsto f \cdot \frac{\theta_1}{x_1^{i_1} y_1^{j_1}} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3},$$

and similarly for the other two mixed cones of Type I.

Proof. As in the statement of the theorem, we will look at the mixed cone corresponding to tridegrees $(-p, q, r)$ with $p \geq 1$ and $q, r \geq 0$. Similar to the proof of Theorem 4.30, we begin by breaking up the trigraded triple complex computing the homology in this mixed cone into $E \oplus Y$ where the elements of the summand E are the domain and image of the red differentials in the following diagram.



That is, the elements of E are the domain and image of a horizontal d^1 differential whose source is a copy of $\mathbb{F}_2$. The homology in this mixed cone is given by the homology of the chain complex

$$E \oplus Y \xrightarrow{d} E \oplus Y,$$

and we can again express the total differential d in terms of this direct sum decomposition as the matrix

$$\begin{bmatrix} d_E & d_E^Y \\ 0 & d_Y \end{bmatrix}.$$

By a similar argument to that given in the proof of Theorem 4.30 in constructing a short exact sequence of chain complexes, we have that the homology in this mixed cone is isomorphic to the homology of the chain complex

$$Y \xrightarrow{d_Y} Y.$$

We now break up Y into a direct sum

$$Y = T_1 \oplus T_2$$

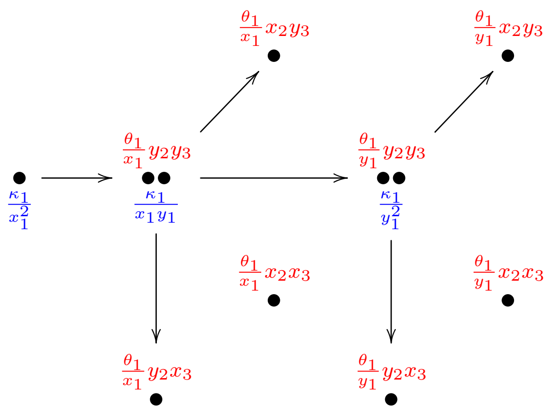
as follows. Note that in each tridegree in Y, the non-zero $\mathbb{F}_2$-modules making up the corresponding triple complex are copies of either $\mathbb{F}_2$ or $\mathbb{F}_2^2$. However, the copies of $\mathbb{F}_2$ were either single copies of $\mathbb{F}_2$ in $E \oplus Y$ or copies of $\mathbb{F}_2$ contained in an $\mathbb{F}_2^2$ in $E \oplus Y$ generated by $(1, 0) \in \mathbb{F}_2^2$. The elements of T_1 at each tridegree are precisely the copies of $\mathbb{F}_2$ generated by the elements $(1, 0)$ in each copy of $\mathbb{F}_2^2$ in Y and the single copies of $\mathbb{F}_2$ in Y that were contained in an $\mathbb{F}_2^2$ in $E \oplus Y$. The elements of T_2 at each tridegree are precisely the copies of $\mathbb{F}_2$ that were single copies of $\mathbb{F}_2$ in $E \oplus Y$ as well as the copies of $\mathbb{F}_2$ generated by the diagonal element $(1,1)$ in an $\mathbb{F}_2^2$ in Y. As in the proof of Theorem 4.30 we can by degree reasons label each element in T_1 by a unique element in

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1^\infty, y_1^\infty)}\{\kappa_1\}$$

and each element in T_2 is a unique homology class in

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1^\infty, y_1^\infty)}\{\theta_1\},$$

and furthermore any element in either of these $\mathbb{F}_2$-modules gives us an element in T_1 or T_2. The following diagram shows the trigraded triple complex $Y = T_1 \oplus T_2$ at tridegree $(-3, 1, 1)$.



As in the proof of Theorem 4.30, we see that the total differential d in the trigraded triple complex Y is given by multiplication by f. □

Notice that each of the three maps in the statement of Theorem 4.31 are neither injective nor surjective, i.e. they each have non-zero kernel and cokernel, unlike in the positive and negative cones. For example, looking at the mixed cone corresponding to tridegrees $(-p, q, r)$ where $p \geq 1$ and $q, r \geq 0$, we see that all elements of the form $\kappa_1 x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}$ are in the kernel of multiplication by f and therefore represent non-zero homology classes. This agrees with our perspective from Section 4.5 of the ring structure of the negative and mixed cones, where the product $\kappa_1 x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}$ of homology classes is non-zero as $\kappa_1 \theta_2 \theta_3 = \Theta$ (by Proposition 4.19) implies in particular that

$$\kappa_1 x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3} \frac{\theta_2}{x_2^{i_2} y_2^{j_2}} \frac{\theta_3}{x_3^{i_3} y_3^{j_3}} = \Theta.$$

Furthermore, notice that the only element in the kernel of multiplication by f in the first map in the statement of Theorem 4.31 at tridegree $(-2, 2, 2)$ is the sum

$$\frac{\kappa_1}{x_1}x_2y_3 + \frac{\kappa_1}{x_1}y_2x_3 + \frac{\kappa_1}{y_1}y_2y_3,$$

which we can therefore identify with the homology class κ_1^2 from the ring structure perspective of Section 4.5, recalling that indeed $\kappa_1^2 x_1 = \kappa_1 x_2 y_3 + \kappa_1 y_2 x_3$ and $\kappa_1^2 y_1 = \kappa_1 y_2 y_3$ by Proposition 4.28. Looking at the cokernel of the first map in the statement of Theorem 4.31, we see for example at tridegree $(-3, 1, 1)$ that $\frac{\theta_1}{y_1}y_2y_3$ is non-zero (i.e. is not in the image of multiplication by f) and that

$$\frac{\theta_1}{y_1}y_2y_3 = \frac{\theta_1}{x_1}x_2y_3 + \frac{\theta_1}{x_1}y_2x_3,$$

which follows since

$$\begin{aligned} d\left(\frac{\kappa_1}{x_1y_1}\right) &= (x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3) \cdot \frac{\theta_1}{x_1y_1} \\ &= \frac{\theta_1}{y_1}y_2y_3 + \frac{\theta_1}{x_1}x_2y_3 + \frac{\theta_1}{x_1}y_2x_3. \end{aligned}$$

From the perspective of Section 4.5, we can identify this homology class with $\kappa_1\theta_1$. Indeed, using Proposition 4.28 we have that

$$\kappa_1\theta_1 = \kappa_1 x_1 \frac{\theta_1}{x_1} = \frac{\theta_1}{x_1}x_2y_3 + \frac{\theta_1}{x_1}y_2x_3,$$

or alternatively

$$\kappa_1\theta_1 = \kappa_1 y_1 \frac{\theta_1}{y_1} = \frac{\theta_1}{y_1}y_2y_3.$$

Finally, we look at the three mixed cones of Type II. For each $i \in \{1, 2, 3\}$, consider the homology class ι_i as defined in Section 4.5. If we take $i = 1$, then we will now be looking at the $\mathbb{F}_2$-module

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_2^\infty, y_2^\infty, x_3^\infty, y_3^\infty)}\{\iota_1\}$$

spanned by elements of the form

$$x_1^{i_1}y_1^{j_1}\frac{\iota_1}{x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}}$$

where $x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}$ is a monomial in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$. Again, not every element of this set will represent a homology class, unlike elements of the $\mathbb{F}_2$-module

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_2^\infty, y_2^\infty, x_3^\infty, y_3^\infty)}\{\theta_2\theta_3\}$$

that we will also be considering.

Theorem 4.32. *The homology in the mixed cone of Type II corresponding to tridegrees $(p, -q, -r)$ with $p \geq 0$ and $q, r \geq 1$ is given by the homology of the chain complex*

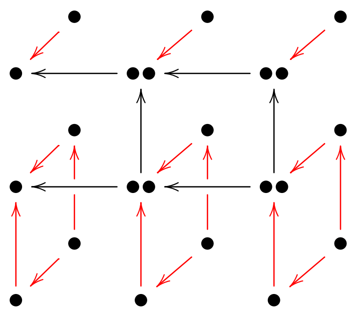
$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_2^\infty, y_2^\infty, x_3^\infty, y_3^\infty)}\{\iota_1\} \xrightarrow{f} \frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_2^\infty, y_2^\infty, x_3^\infty, y_3^\infty)}\{\theta_2\theta_3\}$$

concentrated in two degrees, given by multiplication by the polynomial $f = x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3$. More precisely, the map is given by

$$x_1^{i_1}y_1^{j_1}\frac{\iota_1}{x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}} \mapsto f \cdot x_1^{i_1}y_1^{j_1}\frac{\theta_2\theta_3}{x_2^{i_2}y_2^{j_2}x_3^{i_3}y_3^{j_3}},$$

and similarly for the other two mixed cones of Type II.

Proof. By symmetry, it suffices as in the statement of the theorem to just consider tridegrees $(p, -q, -r)$ where $p \geq 0$ and $q, r \geq 1$. As usual, we first break the trigraded triple complex computing the homology in this mixed cone into a direct sum $F \oplus Z$ where elements in F are the domain and image of any d^2 or d^3 differential whose source is a copy of $\mathbb{F}_2$, i.e. the domain and image of the red differentials in the following diagram.



The homology in this mixed cone is given by the homology of the chain complex

$$F \oplus Z \xrightarrow{d} F \oplus Z$$

where d denotes the total differential, which we can write with respect to this direct sum decomposition as the matrix

$$\begin{bmatrix} d_F & d_F^Z \\ 0 & d_Z \end{bmatrix}.$$

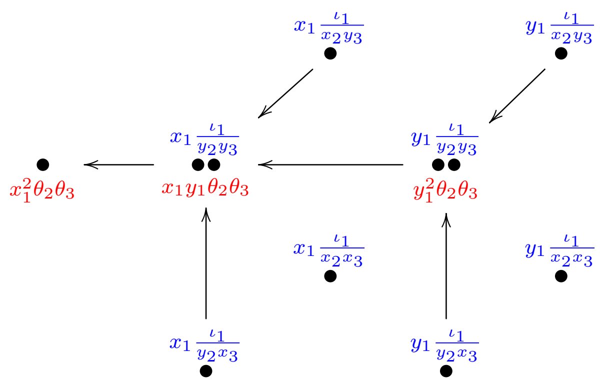
Again, by a similar argument as in the proof of Theorem 4.30, we see that the homology in this mixed cone is isomorphic to the homology of the chain complex

$$Z \xrightarrow{d_Z} Z.$$

We then break up the trigraded triple complex Z into a direct sum $Z = U_1 \oplus U_2$, where the elements of U_1 and U_2 are defined in the same way that the elements of T_1 and T_2 respectively were defined in the proof of Theorem 4.31. By degree reasons, we can identify U_1 and U_2 with

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_2^\infty, y_2^\infty, x_3^\infty, y_3^\infty)}\{\iota_1\} \text{ and } \frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_2^\infty, y_2^\infty, x_3^\infty, y_3^\infty)}\{\theta_2\theta_3\}$$

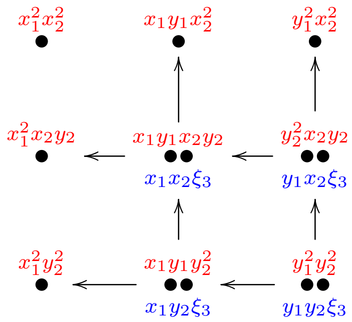
respectively. The following diagram shows the triple complex at tridegree $(2, -2, -2)$ in $Z = U_1 \oplus U_2$.



Again by a similar argument to the proof of Theorem 4.30 we see that the total differential d in the trigraded triple complex Z is given by multiplication by f. □

4.7 The complete Mackey functor structure

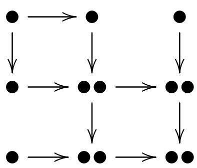
From the perspective of the algebraic description of the top level $\pi_\bigstar^G H\underline{\mathbb{F}_2}$ given in Section 4.6, we will now compute the complete Mackey functor structure of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$. As discussed in Section 4.2, we already know the middle and bottom levels (and the transfer and restriction maps between them) of the homotopy Mackey functors, so it suffices to compute the transfer and restriction maps $\mathrm{Tr}_{H_i}^G$ and $\mathrm{Res}_{H_i}^G$ for each $i \in \{1, 2, 3\}$. In fact, by symmetry it suffices to compute the transfer and restriction maps $\mathrm{Tr}_{H_3}^G$ and $\mathrm{Res}_{H_3}^G$. First, we consider $\pi_*^{H_3}(S^{p\sigma_1+q\sigma_2} \wedge H\underline{\mathbb{F}_2})$ for $p, q \in \mathbb{Z}$. We know that this is isomorphic to $\pi_*^{C_2}(S^{(p+q)\sigma} \wedge H\underline{\mathbb{F}_2})$, but we will focus on the $RO(G)$-grading. If $p, q \geq 0$, then as seen in Section 4.2 this homology at the G/H_3 level is given by the homology of the double complex



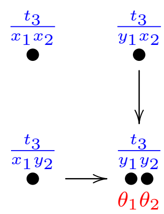
where the above diagram shows bidegree $(2, 2)$, and as usual each $\bullet$ represents a copy of $\mathbb{F}_2$ and each $\bullet\bullet$ represents a copy of $\mathbb{F}_2^2$. Furthermore, the non-zero maps are all codiagonal maps ∇ or $\begin{bmatrix}\nabla\\\nabla\end{bmatrix}$. Hence, by an analogous direct sum decomposition argument as we have done when analysing the top level $\pi_\bigstar^G H\underline{\mathbb{F}_2}$ in Section 4.6, we have that the homology $\pi_*^{H_3}(S^{p\sigma_1+q\sigma_2} \wedge H\underline{\mathbb{F}_2})$ for $p, q \geq 0$ is given by the bidegree (p, q) part of the homology of the chain complex

$$\mathbb{F}_2[x_1, y_1, x_2, y_2]\{\xi_3\} \xrightarrow{f_3} \mathbb{F}_2[x_1, y_1, x_2, y_2]\{1\},$$

where $f_3 = x_1y_2 + y_1x_2$ and ξ_3 is the element $(1, 0) \in \mathbb{F}_2^2$ at bidegree $(1, 1)$. Note that a monomial $x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2}$ in the polynomial ring $\mathbb{F}_2[x_1, y_1, x_2, y_2]\{1\}$ at the G/H_3 level represents the restriction $\mathrm{Res}^G_{H_3}(x_1^{i_1}y_1^{j_1}x_2^{i_2}y_2^{j_2})$ as discussed in the proof of Theorem 4.15, recalling that the restriction map $\mathrm{Res}^G_{H_3}$ is a ring map. Similarly, if we look at the homology $\pi_*^{H_3}(S^{-p\sigma_1-q\sigma_2} \wedge H\underline{\mathbb{F}_2})$ where $p, q \geq 1$ then we are computing the homology of the double complex (looking at bidegree $(-2, -2)$)



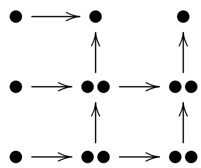
where each non-zero map is either the identity map, or the diagonal maps Δ or $\begin{bmatrix}\Delta & \Delta\end{bmatrix}$. By a similar argument to the proof of Theorem 4.30, the homology of the above double complex is isomorphic to the homology of the double complex



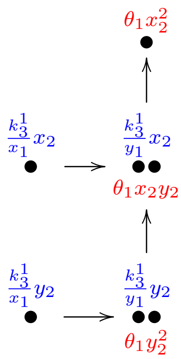
Notice that the diagonal element $(1,1) \in \mathbb{F}_2^2$ at topological degree -4 in the above diagram of bidegree $(-2,-2)$ is precisely $\mathrm{Res}^G_{H_3}(\theta_1\theta_2) = \mathrm{Res}^G_{H_3}(\theta_1)\mathrm{Res}^G_{H_3}(\theta_2)$ which we identify with $\theta_1\theta_2$ (i.e. we identify $\mathrm{Res}^G_{H_3}(\theta_1)$ with θ_1 and $\mathrm{Res}^G_{H_3}(\theta_2)$ with θ_2). If we let t_3 denote the unique non-zero class at bidegree $(-1,-1)$ then the homology $\pi_*^{H_3}(S^{-p\sigma_1-q\sigma_2} \wedge H\underline{\mathbb{F}_2})$ is given by the bidegree $(-p,-q)$ part of the homology of the chain complex

$$\frac{\mathbb{F}_2[x_1,y_1,x_2,y_2]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty)}\{t_3\} \xrightarrow{f_3} \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty)}\{\theta_1\theta_2\},$$

where again $f_3 = x_1y_2 + y_1x_2$. Similarly, if we look at $\pi_*^{H_3}(S^{-p\sigma_1+q\sigma_2} \wedge H\underline{\mathbb{F}_2})$ where $p \geq 1$ and $q \geq 0$, then we are computing the homology of the double complex (looking at bidegree $(-2,2)$)



which is isomorphic to the homology of the double complex



similar to the proof of Theorem 4.31. If we let k_3^1 denote the unique non-zero class at bidegree $(-1,1)$, then identifying $\mathrm{Res}^G_{H_3}(\theta_1)$ with θ_1 as mentioned above, we see that the homology $\pi_*^{H_3}(S^{-p\sigma_1+q\sigma_2} \wedge H\underline{\mathbb{F}_2})$ is given by the bidegree $(-p,q)$ part of the homology of the chain complex

$$\frac{\mathbb{F}_2[x_1,y_1,x_2,y_2]}{(x_1^\infty,y_1^\infty)}\{k_3^1\} \xrightarrow{f_3} \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2]}{(x_1^\infty,y_1^\infty)}\{\theta_1\}.$$

By symmetry, we have that $\pi_*^{H_3}(S^{p\sigma_1-q\sigma_2} \wedge H\underline{\mathbb{F}_2})$ for $p \geq 0$ and $q \geq 1$ is given by the bidegree $(p,-q)$ part of the homology of the chain complex

$$\frac{\mathbb{F}_2[x_1,y_1,x_2,y_2]}{(x_2^\infty,y_2^\infty)}\{k_3^2\} \xrightarrow{f_3} \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2]}{(x_2^\infty,y_2^\infty)}\{\theta_2\}$$

where k_3^2 denotes the unique non-zero class at bidegree $(1,-1)$. Now, recall that

$$\begin{aligned}\pi_k^{H_3}(S^{p\sigma_1+q\sigma_2+r\sigma_3}\wedge H\underline{\mathbb{F}_2}) &\cong \pi_k^{H_3}(S^{p\sigma_1+q\sigma_2+r}\wedge H\underline{\mathbb{F}_2})\\ &\cong \pi_{k-r}^{H_3}(S^{p\sigma_1+q\sigma_2}\wedge H\underline{\mathbb{F}_2})\end{aligned}$$

for all $p,q,r\in\mathbb{Z}$, i.e. if we introduce non-zero multiples of σ_3 then we are just shifting our homology groups to higher or lower degrees via r trivial suspensions, and we use y_3 to keep track of the number of trivial suspensions. That is, if we are looking at $\pi_*^{H_3}(S^{p\sigma_1+q\sigma_2+r}\wedge H\underline{\mathbb{F}_2})$ then we introduce y_3^r to our above expressions for elements of the double complex computing $\pi_*^{H_3}(S^{p\sigma_1+q\sigma_2}\wedge H\underline{\mathbb{F}_2})$.

However, we want to compute the transfer and restriction maps $\mathrm{Tr}_{H_3}^G$ and $\mathrm{Res}_{H_3}^G$ in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$, and therefore as discussed in Section 4.2 we need to look at the H_3-CW structure on $S^{p\sigma_1+q\sigma_2+r\sigma_3}$ (for $p,q,r\geq 0$) giving a triple complex that computes $\pi_*^{H_3}(S^{p\sigma_1+q\sigma_2+r\sigma_3}\wedge H\underline{\mathbb{F}_2})$ where this H_3-CW structure on $S^{r\sigma_3}\simeq S^r$ has two cells in each dimension up to r. This problem can be resolved by giving an explicit chain homotopy equivalence between the triple complex computing $\pi_*^{H_3}(S^{p\sigma_1+q\sigma_2+r\sigma_3}\wedge H\underline{\mathbb{F}_2})$ and the shifted copy of the double complex computing $\pi_*^{H_3}(S^{p\sigma_1+q\sigma_2}\wedge H\underline{\mathbb{F}_2})$. We first look at the positive cone, and although we already know the Mackey functor structure by Theorem 4.15 the perspective of the following theorem is generalisable for computing the Mackey functor structure in the negative and mixed cones.

Theorem 4.33. *The transfer and restriction maps $Tr_{H_3}^G$ and $Res_{H_3}^G$ between the G/H_3 and G/G levels of the positive cone in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ are induced by the maps $T_{H_3}^G$ and $R_{H_3}^G$ defined as follows between the two chain complexes*

$$\begin{array}{ccc}\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]\{\Xi\} & \xrightarrow{f} & \mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]\{1\}\\ & R_{H_3}^G \downarrow\ \uparrow T_{H_3}^G & \\ \mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]\{\xi_3\} & \xrightarrow{f_3} & \mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]\{1\}\end{array}$$

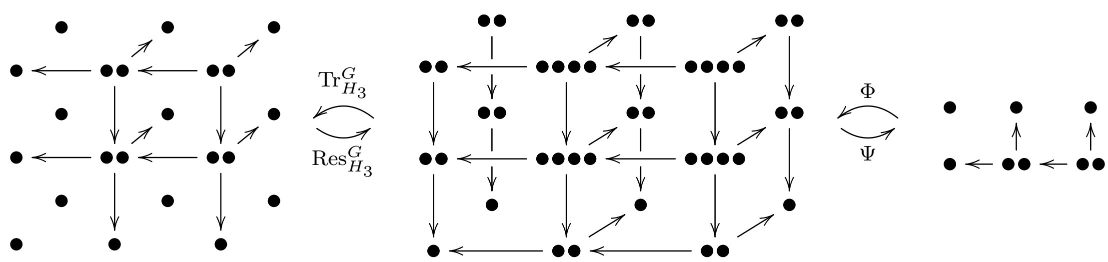
each concentrated in two degrees whose homology computes the G/H_3 and G/G levels of the positive cone. On the generators, we have that

1. $T_{H_3}^G(1)=0$
2. $T_{H_3}^G(\xi_3)=y_1y_2\cdot 1$
3. $R_{H_3}^G(1)=1$
4. $R_{H_3}^G(\Xi)=y_3\cdot\xi_3$.

The maps $T^G_{H_3}$ and $R^G_{H_3}$ are extended linearly to the whole $\mathbb{F}_2$-modules, where $R^G_{H_3}$ applied to any element with a factor of x_3 is zero.

Remark 4.34. In the statement of Theorem 4.33, we see that $R^G_{H_3}(T^G_{H_3}(\xi_3)) = y_1y_2$. This agrees with Proposition 2.15 as the non-trivial element of the Weyl group $W_{H_3}(G)$ sends ξ_3 to y_1y_2. Furthermore, notice that $T^G_{H_3}(R^G_{H_3}(\Xi)) = y_1y_2y_3$, and this does not contradict Proposition 2.23 since in tridegrees (p,q,r) with $r \geq 1$ the maps $T^G_{H_3}$ and $R^G_{H_3}$ are defined as composites of the transfer and restriction maps with non-identity chain homotopy equivalences.

Proof. At each tridegree (p,q,r) where $p,q \geq 0$ and $r \geq 1$ we give explicit chain homotopy equivalences Ψ and Φ between the triple complex computing $\pi^{H_3}_*(S^{p\sigma_1+q\sigma_2+r\sigma_3} \wedge H\underline{\mathbb{F}_2})$ coming from the product G-CW structure on $S^{p\sigma_1} \wedge S^{q\sigma_2} \wedge S^{r\sigma_3}$ and the shift by r trivial suspensions of the double complex computing $\pi^{H_3}_*(S^{p\sigma_1+q\sigma_2} \wedge H\underline{\mathbb{F}_2})$ as follows.



The chain map Φ can be thought of as a diagonal map into the r^{th} level (along the σ_3-direction) of the triple complex computing $\pi^{H_3}_*(S^{p\sigma_1+q\sigma_2+r\sigma_3} \wedge H\underline{\mathbb{F}_2})$, and the chain map Ψ can be thought of as projection onto half of each $\mathbb{F}_2$-vector space on the r^{th} level of the triple complex and zero on the other levels along the σ_3-direction. More precisely, the chain map Φ on a copy of $\mathbb{F}_2$ is given by the diagonal map Δ and on a copy of $\mathbb{F}_2^2$ is given by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

using the notation from Section 4.2. The image of the chain map Φ is precisely what is left over when we take homology in the σ_3-direction in the triple complex at the H_3-level, so we see that the chain map Φ induces an isomorphism in homology. Since Φ is a chain map between chain complexes of vector spaces over the field $\mathbb{F}_2$ that induces an isomorphism in homology, we know by [18, Theorem 10.4.8] that Φ is a chain homotopy equivalence. Letting $T^G_{H_3} = \mathrm{Tr}^G_{H_3} \circ \Phi$ and noting that the transfer maps $\mathrm{Tr}^G_{H_3}$ were computed explicitly in Section 4.2 (for example as the matrix B^T on a copy of $\mathbb{F}_2^4$), we therefore see that $T^G_{H_3}(m \cdot \xi_3) = m \cdot y_1 y_2$ and $T^G_{H_3}(m \cdot 1) = 0$ for any monomial m in $\mathbb{F}_2[x_1, y_1, x_2, y_2, y_3]$.

The chain map Ψ on a copy of $\mathbb{F}_2^2$ in the r^{th} level in the σ_3-direction is projection onto the first component, i.e. is given by the matrix $\begin{bmatrix} 1 & 0 \end{bmatrix}$, and on a copy of $\mathbb{F}_2^4$ in the r^{th} level is given by projection onto the copy of $\mathbb{F}_2^2$ generated by the vectors $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$, i.e. is given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We see that Ψ induces an isomorphism in homology, so by the same argument as above we may deduce that it is a chain homotopy equivalence. Letting $R^G_{H_3} = \Psi \circ \mathrm{Res}^G_{H_3}$ and using for example that the restriction map $\mathrm{Res}^G_{H_3}$ on a copy of $\mathbb{F}_2^2$ is given by the matrix B, we see that $R^G_{H_3}(m \cdot \Xi) = m \cdot y_3 \xi_3$ and $R^G_{H_3}(m \cdot 1) = m \cdot 1$ for any monomial m in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$ such that x_3 does not divide m and is zero otherwise. □

The argument for the transfer and restriction maps in the negative cone is similar, though our chain homotopy equivalences are defined differently, as one would expect.

Theorem 4.35. *The transfer and restriction maps $Tr^G_{H_3}$ and $Res^G_{H_3}$ between the G/H_3 and G/G levels of the negative cone in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ are induced by the maps $T^G_{H_3}$ and $R^G_{H_3}$ defined as follows between the two chain complexes*

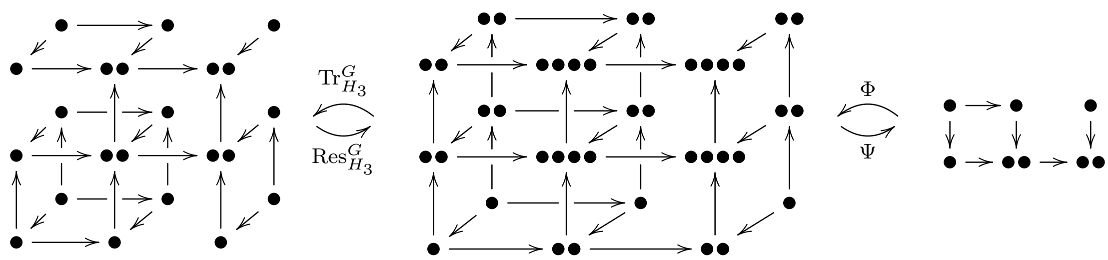
$$\begin{array}{ccc}
\frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty,x_3^\infty,y_3^\infty)}\{\Theta\} & \xrightarrow{f} & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty,x_3^\infty,y_3^\infty)}\{\theta_1\theta_2\theta_3\} \\
 & R^G_{H_3}\Big\downarrow \Big\uparrow T^G_{H_3} & \\
\frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty,y_3^\infty)}\left\{\frac{t_3}{y_3}\right\} & \xrightarrow{f_3} & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty,y_3^\infty)}\left\{\frac{\theta_1\theta_2}{y_3}\right\}
\end{array}$$

each concentrated in two degrees whose homology computes the G/H_3 and G/G levels of the negative cone. On the generators, we have that

1. $T^G_{H_3}\left(\frac{\theta_1\theta_2}{y_3}\right) = y_3 \cdot \theta_1\theta_2\theta_3$
2. $T^G_{H_3}\left(\frac{t_3}{y_3}\right) = \Theta$
3. $R^G_{H_3}(\theta_1\theta_2\theta_3) = 0$
4. $R^G_{H_3}(\Theta) = y_1 y_2 \cdot \frac{\theta_1\theta_2}{y_3}$.

The maps $T^G_{H_3}$ and $R^G_{H_3}$ are extended in the natural way to the whole $\mathbb{F}_2$-modules, where $R^G_{H_3}$ applied to any element involving x_3 is zero.

Proof. At each tridegree $(-p,-q,-r)$ where $p,q,r \geq 1$ we again give explicit chain homotopy equivalences Ψ and Φ between the triple complex computing $\pi^{H_3}_*(S^{-p\sigma_1-q\sigma_2-r\sigma_3} \wedge H\underline{\mathbb{F}_2})$ and the downwards shift by r-many trivial suspensions of the double complex computing $\pi^{H_3}_*(S^{-p\sigma_1-q\sigma_2} \wedge H\underline{\mathbb{F}_2})$.



In this case we can think of the chain map Φ as inclusion into half of each $\mathbb{F}_2$-vector space on the $-r$ level (along the σ_3-direction) of the triple complex computing $\pi^{H_3}_*(S^{-p\sigma_1-q\sigma_2-r\sigma_3} \wedge H\underline{\mathbb{F}_2})$, and the chain map Ψ can be thought of as a codiagonal map on the $-r$ level of the triple complex and zero on the other levels along the σ_3-direction. More precisely, the chain map Φ on a copy of $\mathbb{F}_2$ is given by multiplication by the matrix $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and on a copy of $\mathbb{F}_2^2$ is given by multiplication by the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The image of Φ is precisely what is left over when we take homology in the σ_3-direction in the triple complex, so we see that the chain map Φ induces an isomorphism in homology which implies as in the proof of Theorem 4.33 that Φ is a chain homotopy equivalence. The chain map Ψ is given on a copy of $\mathbb{F}_2^2$ in the $-r$ level of the triple complex by the codiagonal map ∇ and on a copy of $\mathbb{F}_2^4$ in the $-r$ level by multiplication by the matrix

$$A^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

which also induces an isomorphism in homology and is therefore a chain homotopy equivalence. As before, we define $T^G_{H_3} = \mathrm{Tr}^G_{H_3} \circ \Phi$ and $R^G_{H_3} = \Psi \circ \mathrm{Res}^G_{H_3}$. Notice that the matrices used in the definitions of our chain homotopy equivalences in the negative cone are precisely the transposes of the matrices used in the positive cone as given in the proof of Theorem 4.33. Furthermore, if y_3 does not divide a monomial m in $\mathbb{F}_2[x_1, y_1, x_2, y_2, y_3]$ then we see that $T^G_{H_3}(\frac{\theta_1\theta_2}{my_3}) = 0$ which we can therefore identify with $\frac{\theta_1\theta_2\theta_3}{m}y_3$ for degree reasons as this product is indeed zero, and similarly if we are looking at $R^G_{H_3}(\frac{\Theta}{m})$ when m is a monomial in $\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]$ and both x_3 and y_1y_2 do not divide m. □

Finally, we look at the transfer and restriction maps in the six mixed cones. As in Theorems 4.31 and 4.32, it suffices by symmetry to look at one of the three mixed cones of Type I and one of the three mixed cones of Type II. However, when considering each of these mixed cones there will also be two sub-cases as unlike in the positive and negative cones the transfer and restriction maps are not symmetric in the three C_2-subgroups.

Theorem 4.36. *The transfer and restriction maps $Tr^G_{H_3}$ and $Res^G_{H_3}$ between the G/H_3 and G/G levels of the mixed cone of Type I in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ corresponding to tridegrees $(-p, q, r)$ with $p \geq 1$ and $q, r \geq 0$ are induced by the maps $T^G_{H_3}$ and $R^G_{H_3}$ defined as follows between the two chain complexes*

$$\begin{array}{ccc} \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_1^\infty,y_1^\infty)}\{\kappa_1\} & \xrightarrow{f} & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_1^\infty,y_1^\infty)}\{\theta_1\} \\ R^G_{H_3}\big\downarrow & & \big\uparrow T^G_{H_3} \\ \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(x_1^\infty,y_1^\infty)}\{k_3^1\} & \xrightarrow{f_3} & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(x_1^\infty,y_1^\infty)}\{\theta_1\} \end{array}$$

each concentrated in two degrees whose homology computes the G/H_3 and G/G levels of this particular mixed cone. On the generators, we have that

1. *$T^G_{H_3}(\theta_1) = 0$*
2. *$T^G_{H_3}(k_3^1) = y_1y_2 \cdot \theta_1$*
3. *$R^G_{H_3}(\theta_1) = \theta_1$*
4. *$R^G_{H_3}(\kappa_1) = y_3 \cdot k_3^1$.*

The maps $T^G_{H_3}$ and $R^G_{H_3}$ are extended in the natural way to the whole $\mathbb{F}_2$-modules, where $R^G_{H_3}$ applied to any element involving x_3 is zero.

Proof. At each tridegree $(-p, q, r)$ with $p \geq 1$ and $q, r \geq 0$, the chain homotopy equivalences we define between the triple complex computing $\pi^{H_3}_*(S^{-p\sigma_1+q\sigma_2+r\sigma_3} \wedge H\underline{\mathbb{F}_2})$ and the shifted copy of the double complex computing $\pi^{H_3}_*(S^{-p\sigma_1+q\sigma_2} \wedge H\underline{\mathbb{F}_2})$ by r trivial suspensions are defined in the same way as in the proof of Theorem 4.33. We then use the transfer and restriction maps given in Section 4.2 to obtain the expressions for $T^G_{H_3}$ and $R^G_{H_3}$. □

Theorem 4.37. *The transfer and restriction maps $Tr^G_{H_3}$ and $Res^G_{H_3}$ between the G/H_3 and G/G levels of the mixed cone of Type I in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ corresponding to tridegrees $(p, q, -r)$ with $r \geq 1$ and $p, q \geq 0$ are induced by the maps $T^G_{H_3}$ and $R^G_{H_3}$ defined as follows between the two chain complexes*

$$\begin{array}{ccc} \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_3^\infty,y_3^\infty)}\{\kappa_3\} & \xrightarrow{f} & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_3^\infty,y_3^\infty)}\{\theta_3\} \\ & R^G_{H_3} \downarrow \; \uparrow T^G_{H_3} & \\ \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(y_3^\infty)}\left\{\frac{\xi_3}{y_3}\right\} & \xrightarrow{f_3} & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(y_3^\infty)}\left\{\frac{1}{y_3}\right\} \end{array}$$

each concentrated in two degrees whose homology computes the G/H_3 and G/G levels of this particular mixed cone. On the generators, we have that

1. $T^G_{H_3}\left(\frac{1}{y_3}\right) = y_3 \cdot \theta_3$
2. $T^G_{H_3}\left(\frac{\xi_3}{y_3}\right) = \kappa_3$
3. $R^G_{H_3}(\theta_3) = 0$
4. $R^G_{H_3}(\kappa_3) = y_1 y_2 \cdot \frac{1}{y_3}$.

The maps $T^G_{H_3}$ and $R^G_{H_3}$ are extended in the natural way to the whole $\mathbb{F}_2$-modules, where $R^G_{H_3}$ applied to any element involving x_3 is zero.

Proof. At each tridegree $(p, q, -r)$ with $p, q \geq 0$ and $r \geq 1$, the chain homotopy equivalences we define between the triple complex computing $\pi^{H_3}_*(S^{p\sigma_1+q\sigma_2-r\sigma_3} \wedge H\underline{\mathbb{F}_2})$ and the downwards-shifted copy by r trivial suspensions of the double complex computing $\pi^{H_3}_*(S^{p\sigma_1+q\sigma_2} \wedge H\underline{\mathbb{F}_2})$ are defined in the same way as in the proof of Theorem 4.35. We then use the transfer and restriction maps given in Section 4.2 to obtain the expressions for $T^G_{H_3}$ and $R^G_{H_3}$. □

Theorem 4.38. *The transfer and restriction maps $Tr^G_{H_3}$ and $Res^G_{H_3}$ between the G/H_3 and G/G levels of the mixed cone of Type II in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ corresponding to tridegrees $(p, -q, -r)$ with $p \geq 0$ and $q, r \geq 1$ are induced by the maps $T^G_{H_3}$ and $R^G_{H_3}$ defined as follows between the two chain complexes*

$$\begin{array}{ccc} \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_2^\infty,y_2^\infty,x_3^\infty,y_3^\infty)}\{\iota_1\} & \xrightarrow{f} & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_2^\infty,y_2^\infty,x_3^\infty,y_3^\infty)}\{\theta_2\theta_3\} \\ & R^G_{H_3} \downarrow \; \uparrow T^G_{H_3} & \\ \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(x_2^\infty,y_2^\infty,y_3^\infty)}\left\{\frac{k_3^2}{y_3}\right\} & \xrightarrow{f_3} & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(x_2^\infty,y_2^\infty,y_3^\infty)}\left\{\frac{\theta_2}{y_3}\right\} \end{array}$$

each concentrated in two degrees whose homology computes the G/H_3 and G/G levels of this particular mixed cone. On the generators, we have that

1. $T^G_{H_3}\left(\frac{\theta_2}{y_3}\right) = y_3 \cdot \theta_2\theta_3$
2. $T^G_{H_3}\left(\frac{k_3^2}{y_3}\right) = \iota_1$
3. $R^G_{H_3}(\theta_2\theta_3) = 0$
4. $R^G_{H_3}(\iota_1) = y_1y_2 \cdot \frac{\theta_2}{y_3}$.

The maps $T^G_{H_3}$ and $R^G_{H_3}$ are extended in the natural way to the whole $\mathbb{F}_2$-modules, where $R^G_{H_3}$ applied to any element involving x_3 is zero.

Proof. This follows using the same chain homotopy equivalences defined in the proof of Theorem 4.35. □

Theorem 4.39. *The transfer and restriction maps $Tr^G_{H_3}$ and $Res^G_{H_3}$ between the G/H_3 and G/G levels of the mixed cone of Type II in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ corresponding to tridegrees $(-p,-q,r)$ with $p,q \geq 1$ and $r \geq 0$ are induced by the maps $T^G_{H_3}$ and $R^G_{H_3}$ defined as follows between the two chain complexes*

$$\begin{array}{ccc}
\frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty)}\{\iota_3\} & \xrightarrow{f} & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,x_3,y_3]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty)}\{\theta_1\theta_2\} \\
 & R^G_{H_3}\big\downarrow\big\uparrow T^G_{H_3} & \\
\frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty)}\{t_3\} & \xrightarrow{f_3} & \frac{\mathbb{F}_2[x_1,y_1,x_2,y_2,y_3]}{(x_1^\infty,y_1^\infty,x_2^\infty,y_2^\infty)}\{\theta_1\theta_2\}
\end{array}$$

each concentrated in two degrees whose homology computes the G/H_3 and G/G levels of this particular mixed cone. On the generators, we have that

1. $T^G_{H_3}(\theta_1\theta_2) = 0$
2. $T^G_{H_3}(t_3) = y_1y_2 \cdot \theta_1\theta_2$
3. $R^G_{H_3}(\theta_1\theta_2) = \theta_1\theta_2$
4. $R^G_{H_3}(\iota_3) = y_3 \cdot t_3$.

The maps $T^G_{H_3}$ and $R^G_{H_3}$ are extended in the natural way to the whole $\mathbb{F}_2$-modules, where $R^G_{H_3}$ applied to any element involving x_3 is zero.

Proof. This follows using the same chain homotopy equivalences defined in the proof of Theorem 4.33. □

4.8 Applying the Bockstein spectral sequence

In this section we explain how we can use our knowledge of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ to understand $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$ using Bockstein spectral sequences. For simplicity, we will focus on the positive cone and we begin by looking at the top level. Given a fixed tridegree (p,q,r) with $p,q,r \geq 0$, we have a Bockstein spectral sequence

$$\pi^G_*(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_2})[v_0] \Rightarrow \pi^G_*(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{Z}_2^\wedge}) \qquad (4.1)$$

corresponding to the unrolled exact couple

$$\xymatrix{ \cdots \ar[r] & \pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{Z}}) \ar[rr]^-{2} & & \pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{Z}}) \ar[rr]^-{2} \ar[dl] & & \pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{Z}}) \ar[r] \ar[dl] & \cdots \\ & & \pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{F}_2}) \ar[ul] & & \pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{F}_2}) \ar[ul] & & }$$

induced by the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. In particular, we have that $E^1_{s,*} = \pi_s^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3} H\underline{\mathbb{F}_2})[v_0]$ and the d^1 differential is given by $d^1(x) = \beta(x)v_0$ where β is the Bockstein homomorphism, which decreases degree by one, and d^1 increases the v_0-degree by one. More generally, the d^r differential increases v_0-degree by r, and torsion of order 2^k is encoded by a v_0-tower of height k. Further detail on the Bockstein spectral sequence can be found in [25, Chapter 10]. Note that we have similar Bockstein spectral sequences for the G/H_i levels of $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ and $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$ for $i \in \{1, 2, 3\}$, as well as of course the G/e levels.

Theorem 4.40. *The Bockstein spectral sequence (4.1) collapses to the* E^2*-page.*

Proof. By the construction of the Bockstein spectral sequence, it suffices to show that the Bockstein homomorphism β is exact. First, recall by Theorem 4.14 that the positive cone in $\pi_\bigstar^G H\underline{\mathbb{F}_2}$ is given by the ring

$$\frac{\mathbb{F}_2[x_1, y_1, x_2, y_2, x_3, y_3]}{(x_1y_2y_3 + y_1x_2y_3 + y_1y_2x_3)}.$$

Now, observe that $\beta(y_i) = x_i$ and $\beta(x_i) = 0$ for each $i \in \{1, 2, 3\}$. Indeed, the singly-graded chain complex of Mackey functors computing $\underline{\pi}_*(\Sigma^{\sigma_i} H\underline{\mathbb{Z}})$ is given by

$$\xymatrix{ \mathbb{Z} \ar@/_/[d]_{1} & \mathbb{Z} \ar[l]_-{2} \ar@/_/[d]_{\Delta} \\ \mathbb{Z}[G/G] \ar@/_/[u]_{2} \ar@/_/[d]_{1} & \mathbb{Z}[G/H_i] \ar[l]_-{\nabla} \ar@/_/[u]_{\nabla} \ar@/_/[d]_{1} \\ \mathbb{Z}[G/G] \ar@/_/[u]_{2} & \mathbb{Z}[G/H_i] \ar[l]_-{\nabla} \ar@/_/[u]_{2} } \qquad \xymatrix{ \mathbb{Z} \ar@/_/[d]_{1} & \mathbb{Z} \ar[l]_-{2} \ar@/_/[d]_{1} \\ \mathbb{Z} \ar@/_/[u]_{2} \ar@/_/[d]_{1} & \mathbb{Z} \ar[l]_-{2} \ar@/_/[u]_{2} \ar@/_/[d]_{\Delta} \\ \mathbb{Z}[G/G] \ar@/_/[u]_{2} & \mathbb{Z}[G/H_i], \ar[l]_-{\nabla} \ar@/_/[u]_{\nabla} }$$

where on the left we are looking at the G/e, G/H_i and G/G levels of the chain complex of Mackey functors and on the right we are looking at the G/e, G/H_j and G/G levels for any $j \neq i$. In particular, the chain complex computing $\pi_*^G(\Sigma^{\sigma_i} H\underline{\mathbb{Z}})$ is given by $\mathbb{Z} \xleftarrow{2} \mathbb{Z}$, whereas the chain complex computing $\pi_*^G(\Sigma^{\sigma_i} H\underline{\mathbb{F}_2})$ is given by $\mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2$. To compute $\beta(y_i)$, we lift y_i to the integral chain $1 \in \mathbb{Z}$ which maps under the differential to $2 \in \mathbb{Z}$ and thus after dividing by 2 we are left with the integral chain $1 \in \mathbb{Z}$. Finally, after reducing mod 2 we get that $\beta(y_i) = x_i$. Since x_i is in degree 0 and the chain complexes are concentrated in degrees 0 and 1 we have that $\beta(x_i) = 0$. Since we have computed the Bockstein homomorphism β on the generators and we know that β is a derivation (see [26, Section 3.E]), we have therefore completely determined β in the positive cone.

Now, consider an arbitrary class in the positive cone represented by a monomial $m = x_1^{i_1} y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3}$ that is not in the top degree in its corresponding tridegree $(i_1+j_1, i_2+j_2, i_3+j_3)$, so we can assume without loss of generality that $i_1 \geq 1$. Suppose first that j_1, j_2 and j_3 are all even. Then, by writing

$$x_1^{i_1} y_1^{j_1} x_2^{i_2} y_2^{j_2} x_3^{i_3} y_3^{j_3} = x_1^{i_1} y_1^{\frac{j_1}{2}} y_1^{\frac{j_1}{2}} x_2^{i_2} y_2^{\frac{j_2}{2}} y_2^{\frac{j_2}{2}} x_3^{i_3} y_3^{\frac{j_3}{2}} y_3^{\frac{j_3}{2}}$$

and using that β is a derivation determined by $\beta(y_i) = x_i$ and $\beta(x_i) = 0$ for all $i \in \{1, 2, 3\}$, we see that $\beta(m) = 0$. Thus, we need to show that m is in the image of β. However, note that

$$\beta\left(\frac{y_1}{x_1} m\right) = \beta(y_1)\frac{m}{x_1} + y_1 \beta\left(\frac{m}{x_1}\right) = x_1 \frac{m}{x_1} = m,$$

where the second term in the first equality is zero by a similar argument to why $\beta(m) = 0$ using that j_1, j_2 and j_3 are all even. Next, suppose that at least one but at most two of j_1, j_2 and j_3 are odd. Let $k, \ell \in \{1, 2, 3\}$ be such that only j_k and j_ℓ are odd, and let

$$y_{k\ell} = \begin{cases} y_k & \text{if } k = \ell, \\ y_k y_\ell & \text{otherwise.} \end{cases}$$

Then, we have that

$$\beta(m) = \beta(y_{k\ell})\frac{m}{y_{k\ell}} + y_{k\ell} \beta\left(\frac{m}{y_{k\ell}}\right) = \beta(y_{k\ell})\frac{m}{y_{k\ell}},$$

which is non-zero as it is not in the ideal $(x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3)$ since $\beta(y_{k\ell})$ is a sum of at least one and at most two distinct monomials. Finally, suppose that j_1, j_2 and j_3 are all odd. Then, we see that

$$\beta(m) = \beta(y_1 y_2 y_3)\frac{m}{y_1 y_2 y_3} + y_1 y_2 y_3 \beta\left(\frac{m}{y_1 y_2 y_3}\right) = (x_1 y_2 y_3 + y_1 x_2 y_3 + y_1 y_2 x_3)\frac{m}{y_1 y_2 y_3}$$

which is zero, so we again need to show that m is in the image of β. However, we see that

$$\begin{aligned} m &= x_1y_2y_3\frac{m}{x_1y_2y_3} \\ &= (y_1x_2y_3 + y_1y_2x_3)\frac{m}{x_1y_2y_3} \\ &= (x_2y_3 + y_2x_3)\frac{y_1m}{x_1y_2y_3} \\ &= \beta\left(y_2y_3 \cdot \frac{y_1m}{x_1y_2y_3}\right), \end{aligned}$$

where the last equality follows since $\frac{y_1m}{x_1y_2y_3}$ contains only even powers of y_1, y_2 and y_3. □

By the result of Theorem 4.40 we know that all torsion has order 2, i.e. there is no torsion of order 2^k for $k > 1$ as the E^∞-term contains only v_0-towers of height 1. Since $\beta(y_1^{j_1}y_2^{j_2}y_3^{j_3}) = 0$ if and only if j_1, j_2 and j_3 have the same parity, it follows that we get a $\mathbb{Z}$ at the top degree in each tridegree (p, q, r) if p, q and r have the same parity and otherwise we get 0. In all other degrees, the non-zero homology groups will be various powers of $\mathbb{Z}/2$ given by looking at the rank of β in each degree. We can in fact write down a Poincaré series for the 2-torsion given by dividing the Poincaré series of Theorem 4.2 by $x + 1$ (after subtracting any term contributing a $\mathbb{Z}$ in homology), using that at each tridegree β gives an exact sequence of $\mathbb{F}_2$-vector spaces (with dimensions given by the Poincaré series of Theorem 4.2) as well as the series expansion of $\frac{1}{x+1}$.

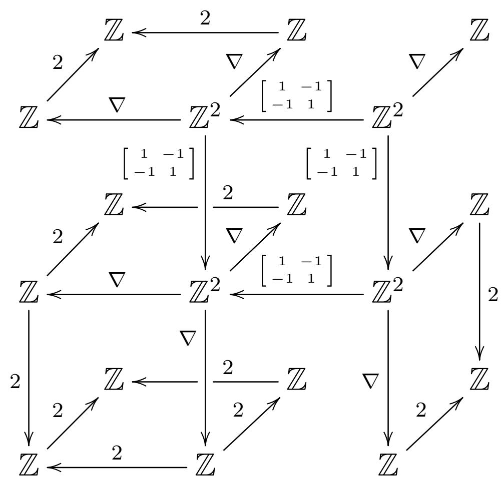
Remark 4.41. Since the Bockstein spectral sequence converges to the homology with coefficients in $\mathbb{Z}_2^\wedge$ (rather than with coefficients in $\mathbb{Z}$), we are using in the above that the homology groups with integer coefficients are finitely-generated, which follows since these are given by the homology of the triple complex obtained by tensoring the cellular chain complexes of $\mathbb{Z}[G]$-modules for our explicit G-CW structures on $S^{p\sigma_1}$, $S^{q\sigma_2}$ and $S^{r\sigma_3}$ given in Section 4.2. Furthermore, we are using that there is no p-torsion for any odd prime p, and this can be seen by running the Bockstein spectral sequence induced by the short exact sequence $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p \to 0$ after computing the homology $\pi_*^G(\Sigma^{p\sigma_1+q\sigma_2+r\sigma_3}H\underline{\mathbb{F}_p})$ which is zero except possibly at the top degree depending on p, q and r as for example the map $\mathbb{F}_p \xrightarrow{2} \mathbb{F}_p$ is an isomorphism.

Note that by the result of Theorem 4.15, for example that the G/H_1 level of the positive cone in $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ is given by the ring

$$\frac{\mathbb{F}_2[y_1, x_2, y_2, x_3, y_3]}{(x_2y_3 + y_2x_3)},$$

a similar argument to the proof of Theorem 4.40 can be used to show that the Bockstein spectral sequence on the G/H_i levels for each $i \in \{1, 2, 3\}$ collapses to the E^2-page, and moreover since $\beta(y_i) = x_i$ and $\beta(x_i) = 0$ we see that β commutes with the restriction maps. Therefore, we can in fact use the Bockstein spectral sequence on the level of Mackey functors (together with Proposition 2.15) to help deduce $\underline{\pi}_\bigstar H\underline{\mathbb{Z}}$ from $\underline{\pi}_\bigstar H\underline{\mathbb{F}_2}$ in the positive cone. We claim that similar results involving the Bockstein spectral sequence hold for the negative and mixed cones using the algebraic descriptions from Sections 4.6 and 4.7, but we do not explore this here. Alternatively, we can compute the homology at the top level with integer coefficients by iteratively using the spectral sequence of a double complex (as in Section 4.3) to compute the homology of the analogous triple complex with integer coefficients at each tridegree $(p, q, r) \in \mathbb{Z}^3$ as discussed in Section 4.2.

Example 4.42. Suppose that we want to compute $\pi_*^G(\Sigma^{2\sigma_1+\sigma_2+2\sigma_3} H\underline{\mathbb{Z}})$ directly, without using the Bockstein spectral sequence. That is, we want to compute the homology of the triple complex



As in the proof of Theorem 4.2, we will iteratively use the spectral sequence of a double complex. In particular, we run the spectral sequence converging to the homology with respect to $d^1 + d^2$ where we first take homology with respect to d^1, giving us the following E^1-page (noting that we draw the horizontal cross sections of the above triple complex with increasing degree in the σ_3-direction going left to right):

$$\begin{array}{ccc} \mathbb{Z}/2 & 0 & \mathbb{Z} \\ & & \uparrow{\scriptstyle 2} \\ \mathbb{Z}/2 & 0 & \mathbb{Z} \end{array} \qquad \begin{array}{ccc} \mathbb{Z}/2 & 0 & \mathbb{Z} \\ & & \uparrow{\scriptstyle 2} \\ 0 & 0 & \mathbb{Z} \end{array} \qquad \begin{array}{ccc} \mathbb{Z}/2 & 0 & \mathbb{Z} \\ & & \uparrow{\scriptstyle 2} \\ 0 & 0 & \mathbb{Z} \end{array}$$

Therefore, after taking homology with respect to d^2 we get the E^2-page

$$\begin{array}{ccc} \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \\ \mathbb{Z}/2 & 0 & 0 \end{array} \qquad \begin{array}{ccc} \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \\ 0 & 0 & 0 \end{array} \qquad \begin{array}{ccc} \mathbb{Z}/2 & 0 & \mathbb{Z}/2 \\ 0 & 0 & 0 \end{array}$$

Notice that there are no higher differentials, so the spectral sequence converging to the homology with respect to $d^1 + d^2$ collapses on the E^2-page. Hence, since there are no non-zero differentials in the σ_3-direction on the above E^2-page it follows that the Poincaré series of the 2-torsion is given by $1 + 2x + 2x^2 + x^3 + x^4$, which indeed is precisely what we get by dividing the Poincaré series of Theorem 4.2 at tridegree $(2, 1, 2)$ by $x + 1$.

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